A CLASS OF BANACH JORDAN ALGEBRAS

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ABSTRACT

In this thesis, we study JB*-algebras, the Jordan algebra analogues of B*-algebras. Our results depend heavily on a recent Gelfand-Neumark theorem for unital JB-algebras obtained by Alfsen, Shultz and Størmer.

The first Chapter, which contains the background material we require, is essentially introductory. The main result of the second Chapter is an analogue of the Vidav Palmer theorem for complex unital Banach Jordan algebras, which we use together with a strong version of the Russo-Dye theorem to derive many of the properties of unital JB*-algebras. In the third Chapter, we consider the problem of adjoining a unit to a JB*-algebra, and give some of the theory of the Jordan algebra analogues of von Neumann algebras, the unital JB*-algebras which are Banach dual spaces.

In the fourth Chapter, we give an algebraic characterisation of the isometries of a unital JB*-algebra onto itself. From this, we deduce an algebraic characterisation of the Hermitian operators on a unital JB*-algebra, and give a large class of Hermitian operators whose squares are not Hermitian. We also investigate the links between JB*-algebras and bounded symmetric homogeneous domains. Finally, the fifth Chapter contains some conjectures and open problems.

CONTENTS

Introduction

The axioms for a Jordan algebra first appear in the work of Jordan, von Neumann and Wigner[43], who were seeking a suitable mathematical framework for the theory of quantum mechanics, but subsequent developments in this area did not follow their approach. Although they had considered only real, finite dimensional Jordan algebras, it was an easy step to transfer the axioms and consider Jordan algebras over an arbitrary field, and later on, an arbitrary ring. The algebraic theory of Jordan algebras has steadily developed since then, one stimulus being the unexpected connections between Lie and Jordan algebras.

By contrast, the theory of Banach Jordan algebras has received little attention, mainly due to the lack of representation theory. The first significant results for Banach Jordan algebras were for those which were actually Jordan subalgebras of Banach algebras, and in [84], Topping gave a satisfactory classification theory for a weak operator closed real Jordan subalgebra A of self-adjoint operators on a complex Hilbert space, which agreed with the usual one when A is the self-adjoint part of a von Neumann algebra. Further work in this area was done by Effros and Størmer [80], [81].

However the restriction to Banach Jordan algebras which are Jordan subalgebras of Banach algebras was a retreat from the initial work of Jordan, von Neumann and Wigner, and so, it was a major advance when Alfsen, Shultz and Størmer introduced and developed a theory for JB-algebras in [5] . JB-algebras are a class of real Banach Jordan algebras, and they include the JW-algebras studied by Topping [84] , and the finite dimensional, formally real Jordan algebras.

Despite the success of this theory, JB-algebras are real Banach Jordan algebras, while most Banach algebra theory is concerned with complex Banach algebras. It was possibly with this in mind that Kaplansky introduced and pointed out the potential importance of "Jordan C*-algebras" at the Edinburgh Mathematical Society Colloquium in St. Andrews in 1976. This choice of name was perhaps unfortunate, as the name "Jordan C*-algebra" is what Kadison in [47] called the self-adjoint part of a C*-algebra, so we shall call Kaplansky's "Jordan C*-algebras", JB*-algebras. This name fits with B*-algebras being the abstract version of C*-algebras, as JB*-algebras are complex Banach Jordan algebras with an involution, which satisfy a condition which ties the norm and involution structure together,and are the abstract version of JC*-algebras, the closed self-adjoint Jordan subalgebras of the bounded linear operators on a complex Hilbert space. This thesis is concerned with some of the properties of JB*-algebras.

Both JB-algebras and JB*-algebras have important applications to other topics in functional analysis, as the following examples show:

(i) Koecher[54], and Vinberg [87],[88], showed that the finite dimensional formally real Jordan algebras are in one to one correspondance with finite dimensional transitively homogeneous self-dual cones. In recent work on von Neumann algebras [8],[25], orientable facially homogeneous self-dual cones played an important role, and the problem arose of classifying all such cones. Partial solutions, which link the cone to a JB-algebra have been given in [9] and [10]. JB-algebras have also connections with other structures encountered in quantum mechanics [61], [73],[75],[26],[4].

(ii) In 1935, Cartan[20]classified all finite dimensional bounded symmetric homogeneous domains, using the theory of Lie groups and Lie algebras. In 1969, Koecher [55] gave a different approach, which emphasised the importance of Jordan triple systems, and it was this approach which Kaup [50]generalised. We shall show in Chapter 4, that the open unit ball of a unital JB*-algebra is a bounded symmetric homogeneous domain, and show how JB*-algebras arise from some of the Jordan triple systems.

We now summarise the contents of this thesis. There are five chapters, each having an introduction giving the contents of that chapter. Without anticipating many of the definitions, we cannot go into such detail here. Chapter 1 is essentially introductory, and it contains the background material we shall require. We cover some of the algebraic theory of Jordan algebras, and analytic theory of Banach Jordan algebras, introduce the JB-algebras of Alfsen, Shultz and Størmer, and conclude the chapter with results about differentiability and homogeneous domains. The major result of Chapter 2 is a Vidav-Palmer theorem for JB*-algebras, and for this, we have to extend results on numerical range from Banach algebras to Banach Jordan algebras. In addition, we prove a strong version of the Russo-Dye theorem for JB*-algebras.

Chapter 3 is concerned with two main problems; the first is adjoining a unit to a JB*-algebra, and the second is the theory of JB*-algebras which are Banach dual spaces. These correspond to von Neumann algebras in the associative case, and we only make a start on the theory. In Chapter 4, we prove Kaplansky's conjecture that a linear isometry between unital JB*-algebras which takes the unit of the first onto the unit of the second is a Jordan homomorphism, and in fact, a more general problem about arbitrary linear isometries between unital JB*-algebras. To do this, we have to show that the open unit ball of a unital JB*-algebra is a bounded symmetric homogeneous domain. Finally, in Chapter 5, we give some open problems and conjectures.

As Kaplansky expected in[49], since the work of Alfsen, Shultz and Størmer, there has been a flurry of activity on Banach Jordan algebras, and JB and JB*-algebras in particular. So, some of the work appearing in this thesis has either been published, or submitted for publication [94],[95],[96],[97]. In addition, some results have been proved independently by other authors [30],[78],[93]. However, as in the rest of the thesis, the work presented in all these areas is my own original work, unless specific mention is made to the contrary.

We conclude this introduction with a few remarks on the material and notation which we assume. Throughout, we assume and use the standard theory of functional analysis. In addition, we state, without proof, many results on the algebraic theory of Jordan algebras, and the main results of Alfsen, Shultz and Størmer on JB-algebras. We shall not always give the original reference to well known results.

We let $\mathbb{R}$ and $\mathbb{C}$ denote respectively the real and complex fields. The symbol $\mathbb{F}$ will be used to denote a field which is either $\mathbb{R}$ or $\mathbb{C}$. We let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{N}$ = the set of positive integers, and $\mathbb{P} = \{0\} \cup \mathbb{N}$, the non-negative integers. Further, we let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and, for z in $\mathbb{C}$, we let Rez denote $\frac{1}{2}(z+\bar{z})$.

By a linear space, we shall mean a linear space over $\mathbb{F}$. Let X and Y be normed linear spaces. We denote by X' the dual of X, that is, the Banach space of continuous linear functionals on X, and by X_1 the closed unit ball of X. We denote by $B(X,Y)$ the normed linear space of continuous linear operators from X into Y, with the usual supremum norm, and write $B(X)$ in place of $B(X,X)$. Given $T \in B(X,Y)$, the adjoint of T, denoted by T' is the element of $B(Y',X')$ defined by

$$(T'(f)(x) = f(T(x))$$

for $f \in Y'$ and $x \in X$. (We do not use the more common T^* for the adjoint of T, as we shall frequently require the symbol T^* to denote the image of T under an involution $*$). If S is any map from X into Y, we let $\mathrm{Ker}S = \{x : Sx = 0\}$ and $\mathrm{Im}S = \{y \in Y : \text{there exists } x \text{ in } X \text{ with } Sx = y\}$.

If E is a non-empty set of a normed linear space X, we let $\mathrm{co}E$ and $\overline{\mathrm{co}}E$ denote respectively the convex hull and the closed convex hull of E, and we let $\mathrm{Sp}E$ denote the linear span of E in X. Further, if X is a complex normed linear space, we denote the absolutely convex hull of E by $|\mathrm{co}E|$ and if $\Gamma = \{\alpha_1, \ldots, \alpha_n\}$ is a finite subset of $\mathbb{C}$, we let $\Gamma E = \bigcup_{j=1}^{n} \{\alpha_j e : e \in E\}$.

If M is a separating linear subspace of X', we let $\tau(X,M)$ denote the weak topology on X induced by M, that is, the weakest topology on X in which every element of M is continuous, see [66]. (We do not use the more common $\sigma(X,M)$ to avoid possible confusion with the spectrum of an element). If no confusion arises, we sometimes identify X with its canonical image in $X'' = (X')'$, so that, for example, $\tau(X',X)$ denotes the weak *-topology on X.

Finally, as is customary, we use the term "the Hahn-Banach theorem" to cover several closely related results given in [66] Chapter 3, Section 1.

CHAPTER 1

This chapter contains much of the background material which we shall require later on, and none of the material is original. We start with the algebraic theory of non-associative algebras, and in particular Jordan algebras, which is mainly taken from Jacobson [43]. We next consider some of the elementary properties of Banach Jordan algebras. These include the properties of the spectrum of an element, due to Devapakkiam [27], the complexification and adjoining a unit, which is similar to the Banach algebra case [14], and a construction due to Arens [7] which extends the product on a Banach Jordan algebra to a product on its double dual.

Then, we present some of the theory of JB-algebras due to Alfsen, Shultz and Størmer. [5]. We prove some of the elementary results in some detail, but only give a broad outline of the methods they use in proving their Gelfand Neumark theorem. We conclude the chapter with results of Harris [40]. These are mainly concerned with Mobius transformations of the open unit ball of certain subspaces of B(H), where H is a complex Hilbert space, and some of their consequences. Finally, we give a simplification, due to Harris, of a result of Segal [74].

1.1 Non-associative algebras.

DEFINITION An algebra A is a linear space over $\mathbb{F}$ with a map $\wedge: A \times A \to A$ called the product in A, such that

(i) $a \wedge (b+c) = a \wedge b + a \wedge c$, $(a+b) \wedge c = a \wedge c + b \wedge c$

(ii) $(\lambda a) \wedge b = a \wedge (\lambda b) = \lambda (a \wedge b)$

for all a, b and c in A and $\lambda \in \mathbb{F}$. A non-zero element e of A is called a unit if, for all a in A,

$$a \wedge e = e \wedge a = a \ ,$$

and A is called an algebra with unit if A has a unit.

Remark If $\mathbb{F} = \mathbb{R}$, A is called a real algebra, and if $\mathbb{F} = \mathbb{C}$, A is called a complex algebra.

DEFINITION. Let $(A, \wedge)$ and $(B, \wedge)$ be algebras over $\mathbb{F}$. A homomorphism of A into B is a linear map of A into B such that, for all a and b in A,

$$G(a \wedge b) = G(a) \wedge G(b).$$

If, in addition, G is one to one and onto, G is called an isomorphism.

DEFINITION. Let $(A, \wedge)$ be an algebra, and let B be a linear subspace of A.

(i) B is a subalgebra of A if $x \wedge y \in B$ whenever $x \in B$ and $y \in B$.

(ii) B is an ideal of A, if $x \wedge y \in B$ and $y \wedge x \in B$ whenever $x \in B$ and $y \in A$.

The following Theorem, [70] p10, is the expected connection between homomorphisms and ideals.

THEOREM 1·1·1. Let $(A, \wedge)$ and $(B, \wedge)$ be algebras over $\mathbb{F}$, I an ideal of A, and G a homomorphism of A into B. Then the quotient space A/I is an algebra with product defined by

$$(a+I) \wedge (b+I) = (a \wedge b) + I$$

for a and b in I, $\ker G$ is an ideal, $\mathrm{Im}G$ is a subalgebra, and G induces an isomorphism M of $A/\mathrm{Ker}G$ onto $\mathrm{Im}G$ defined by

$$M(a+I) = G(a)$$

for a in A.

It is usually necessary to restrict attention to algebras which satisfy additional axioms.

DEFINITION. Let $(A, \wedge)$ be an algebra.

(i) A is associative if, for all a, b and c in A, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

(ii) A is commutative if, for all a and b in A, $a \wedge b = b \wedge a$.

As the associative axiom is very often incorporated into the definition of an algebra, especially in functional analysis, we shall sometimes call an algebra, A, a non-associative algebra to emphasise that A need not be associative.

Apart from the associative algebras, three other classes of non-associative algebras have a well developed theory. These are the following classes.

DEFINITION. Let $(A, \wedge)$ be an algebra.

(i) A is an alternative algebra if $(x \wedge x) \wedge y = x \wedge (x \wedge y)$ and $(x \wedge y) \wedge y = x \wedge (y \wedge y)$ for all x and y in A.

(ii) A is a Jordan algebra if A is commutative and

$$(x \wedge x) \wedge (x \wedge y) = x \wedge ((x \wedge x) \wedge y)$$

for all x and y in A.

(iii) A is a Lie algebra if

$$(x\wedge y) + (y\wedge x) = 0$$

and
$$(x\wedge y)\wedge z + (y\wedge z)\wedge x + (z\wedge x)\wedge y = 0$$

for all x, y and z in A .

Remarks (i) It is customary to omit the multiplication sign $\wedge$ in associative and alternative algebras, with the product just being denoted by juxtaposition. For Jordan algebras, the product is usually denoted by $x \circ y$, while for Lie algebras by $[x,y]$, for x and y in the algebra.

(ii) For x in an algebra $(A,\wedge)$, and $n \in \mathbb{N}$, we define $x^1 = x$ and by induction $x^n = (x^{n-1})\wedge x$, for $n \geq 2$. If $(A,\wedge)$ has a unit e , then we let $x^0 = e$.

(iii) With the above conventions, the definition of, for example, alternative algebras would be $(A,\wedge)$ is alternative if

$$x^2y = x(xy) \quad \text{and} \quad (xy)y = xy^2$$

for all x and y in A . This is the more common formulation of the axioms.

(iv) The property of being associative , alternative , Jordan or Lie is hereditary, that is, a subspace or a quotient space of an associative (respectively, alternative, Jordan, Lie) algebra is an associative (respectively, alternative, Jordan, Lie) algebra.

Let us now give examples of these non-associative algebras.

Example 1·1·1 Let $\mathbb{K}$ be the Cayley algebra over $\mathbb{R}$. Then $\mathbb{K}$ is an 8-dimensional real alternative algebra [70].

<u>Example 1·1·2</u> Let A be an associative algebra, and let S be a subspace of A such that if $a \in S$, then $a^n \in S$ for every $n \in \mathbb{N}$. Then S, with product o, where $aob = \frac{1}{2}(ab+ba)$ for a and b in S is a Jordan algebra, and S is called a Jordan subalgebra of A. A Jordan algebra B, which is isomorphic to such a Jordan algebra (S,o) is called special [43].

<u>Example 1·1·3</u> Let A be an associative algebra, and let L be a subspace of A, such that given a and b are in L, we have $ab - ba \in L$. Then L, with product $[a,b] = ab - ba$, is a Lie algebra.

<u>Example 1·1·4</u> Let M_3^8 denote the algebra of all 3×3 matrices of the form

$$\begin{bmatrix} b_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & b_2 & a_1 \\ a_2 & \bar{a}_1 & b_3 \end{bmatrix}$$

where $a_1, a_2, a_3 \in \mathbb{K}$, $\bar{a}_1$, $\bar{a}_2$ and $\bar{a}_3$ are their conjugates, and $b_1, b_2, b_3 \in \mathbb{R}$. Then, if $a \wedge b$ denotes the usual matrix multiplication, M_3^8, with multiplication $aob = \frac{1}{2}(a \wedge b + b \wedge a)$ is a Jordan algebra. (M_3^8, o) is an exceptional Jordan algebra, that is, it is not a special Jordan algebra [1], [43].

The conjugates of elements of $\mathbb{K}$ are images under an involution. This is defined for an algebra as follows:

DEFINITION. (a) Let A be an algebra over $\mathbb{F}$. A linear involution on A is a map $* : A \to A$ such that

(i) $(a+b)^* = a^* + b^*$,

(ii) $a^{**} = a$,

(iii) $(\lambda a)^* = \bar{\lambda} a^*$,

for all a and b in A and $\lambda \in \mathbb{F}$, where $\bar{\lambda}$ denotes the complex conjugate of λ if $\lambda \in \mathbb{C}$, and $\bar{\lambda} = \lambda$ if $\lambda \in \mathbb{R}$.

An element a such that $a^* = a$ is called self-adjoint.

(b) Let $(A, \wedge)$ be an algebra over $\mathbb{F}$. An involution on A is a linear involution on A such that, for all a and b in A, we have

$$(a \wedge b)^* = b^* \wedge a^* .$$

We now give an example which generalises Example 1·1·4.

Example 1·1·5 Let B be a real algebra with involution $*$ and unit. Let $M_n(B)$ be the algebra of $n \times n$ matrices with entries in B under the usual multiplication. Let $\dagger$ denote the involution defined on $M_n(B)$ by $[m_{ij}]^\dagger = [m_{ij}^*]^t$ where a^t denotes the transpose of the matrix a, and let $S_n(B)$ denote the self-adjoint elements of $(M_n(B), \dagger)$. Given a, b in $S_n(B)$, we define $a \circ b = \frac{1}{2}(a \wedge b + b \wedge a)$. If $(S_n(B), \circ)$ is a Jordan algebra, it is called a Jordan matrix algebra.

We shall mainly be concerned with complex algebras with involution, and the following result gives a characterisation of when it is possible to put an involution on an algebra.

LEMMA 1·1·2. (i) Let A be a complex algebra with involution, and let B denote the set of self-adjoint elements of A.
Then B is a real linear subspace of A such that $A = B \oplus iB$, and such that, whenever a and b are in B, $a \wedge b + b \wedge a$ and $i(a \wedge b - b \wedge a)$ are in B.

(ii) Let A be a complex algebra and B a real linear subspace such that $A = B \oplus iB$, and such that, whenever a and b are in B,

$a \wedge b + b \wedge a$ and $i(a \wedge b - b \wedge a)$ are in B. Then there is an involution $*$ on A defined by

$$(a+ib)^* = a - ib$$

for a and b in B, such that the set of self-adjoint elements of $(A,{}^*)$ is B.

DEFINITION. Let A be a complex linear space, and B a real linear subspace of A such that $A = B \oplus iB$. Given $a \in A$, the unique decomposition of a into $h + ik$ where h and k are in B is called the standard decomposition of a. Further, the map defined by

$$(x+iy)^* = x - iy$$

for x and y in B is called the natural linear involution on A. If A is an algebra, and $*$ is an involution, $*$ is called the natural involution on A.

One of the major handicaps in the theory of non-associative algebra is the lack of natural representations.

DEFINITION. Let $(A, \wedge)$ be an algebra. Given $b \in B$, we define $L_b : A \to A$ by

$$L_b(a) = b \wedge a .$$

As $(A, \wedge)$ is an algebra, L_b is a linear map for all b in A, but $b \to L_b$ is a homomorphism if and only if A is associative.

For the remainder of the section, we shall give the elementary theory of Jordan algebras, relying mainly on [43]. A Jordan algebra $(A, \circ)$ has an important trilinear map defined by

$$\{a,b,c\} = (a \circ b) \circ c - (a \circ c) \circ b + (b \circ c) \circ a$$

for all a, b and c in A . It is immediate that

$$\{a,b,c\} = \{c,b,a\}$$

for all a, b and c in A , and that, if A has a unit,

$$\{a,b,1\} = \{a,1,b\} = \{1,a,b\} = a \circ b$$

for all a and b in A . Given a and b in A , we define a linear operator $U_{a,b} : A \to A$ by

$$U_{a,b}(x) = \{a,x,b\} ,$$

and abbreviate $U_{a,a}$ to U_a. Then

$$U_a = 2(L_a)^2 - L_{a^2} .$$

As multiplication in A is not associative, we cannot always "remove brackets" in formulae, and obtain identities which are still valid. However one way to get identities which are valid in all Jordan algebras is to obtain linearisations of the defining identities. As an example, the following formulae are valid in any Jordon algebra $(A,\circ)$.

$$(a\circ b)\circ(c\circ d) + (a\circ d)\circ(b\circ c) + (a\circ c)\circ(b\circ d) =$$
$$(a\circ(c\circ d))\circ b + (a\circ(b\circ c))\circ d + (a\circ(b\circ d))\circ c$$

$$(a\circ(b\circ c))\circ d + (a\circ(b\circ d))\circ c + (a\circ(c\circ d))\circ b =$$
$$((a\circ b)\circ c)\circ d + ((a\circ d)\circ c)\circ b + a\circ((b\circ d)\circ c)$$

$$\{a,b,c\}\circ d = \{a\circ d,b,c\} - \{a,b\circ d,c\} + \{a,b,c\circ d\}$$

$$[[L_c,L_b],L_d] = L_{(b\circ d)\circ c} - L_{b\circ(d\circ c)} ,$$

where $[T,S] = TS - ST$ for linear operators S and T on A . These are given in [43] pp33-37, and are all that is required to give the following results, [43] Theorems 1·5 and 1·6.

LEMMA 1·1·3. Let $(A,\circ)$ be a Jordan algebra over $\mathbb{F}$, without a unit. Then $(A\oplus\mathbb{F}, \wedge)$ is a Jordan algebra over $\mathbb{F}$ with unit $(0,1)$ where

$$(a,\mu) \wedge (b,\lambda) = (a\circ b + \mu b + \lambda a,\mu\lambda)$$

LEMMA 1·1·4. Let (B,o) be a real Jordan algebra. The complexification $(A,\wedge)$ of (B,o) is the set $B \times B$ with addition, scalar multiplication and product defined for all a, b, c and d in A and $\alpha,\beta \in \mathbb{R}$ by

$$(a,b) + (c,d) = (a+c, b+d) ,$$
$$(\alpha+i\beta)(a,b) = (\alpha a-\beta b, \alpha b+\beta a) ,$$
$$(a,b) \wedge (c,d) = (aoc-bod, aod+boc) .$$

Then $(A,\wedge)$ is a complex Jordan algebra with natural involution * whose set of self-adjoint elements is B . In addition, if B has a unit 1, $(1,0)$ is a unit for A .

The process of deriving identities by linearisation is, however, of limited application, and to get more information, it is necessary to use deeper Theorems on Jordan algebras. It is convenient to introduce the following notation first.

Notation If A is a Jordan algebra, and $x_1,\ldots, x_r \in A$, we denote by $Q(x_1,\ldots, x_r)$ the Jordan algebra generated by $x_1,\ldots, x_r$.

THEOREM 1·1·5. Let A be a Jordan algebra and $a \in A$. Then if k and $m \in \mathbb{N}$,

$$a^{k+m} = (a^k)o(a^m) .$$

and so $(Q(a),o)$ is an associative commutative algebra. If A has a unit, $(Q(1,a),o)$ is also an associative commutative algebra.

A proof of Theorem 1·1·5 may be found in [43] p36. It shows in particular that $Q(a)$ is special ; the next result improves this, [43] p48.

THEOREM 1·1·6. (Shirshov-Cohn). Let A be a Jordan algebra, and let $a, b \in A$. Then $Q(a,b)$ is special, and if in addition A has a unit, $Q(1,a,b)$ is also special.

On the other hand, to prove a Jordan algebra is non-special is not easy in general, but there are identities called "s-identities" which are known to hold for all special Jordan algebras, but not all Jordan algebras. Although Albert [1] had previously proved M_3^8 was not special, Glennie [33] produced an s-identity

$$4\{\{z,\{x,y,x\},z\}y,(x\circ z)\} - 2\{z,\{x,\{y,x\circ z,y\},x\},z\}$$
$$- 4\{(x\circ z),y,\{x,\{z,y,z\},x\}\} + 2\{x,\{z,\{y,x\circ z,y\},z\},x\},$$

which held in all special Jordan algebras but not in M_3^8. We note it is a three variable identity of degree 8, and remark that there are no s-identities of degree ≤ 7 in three variables nor any s-identities in at most 5 variables which are of first degree in all of these. Another Theorem in the same area is the following, due to Macdonald, which is proved in [43] p41.

THEOREM 1·1·7. If a three variable identity is of degree at most one in one of the variables, and holds in all special Jordan algebras, then it holds in all Jordan algebras.

As an example of the use of this Theorem, let A be a Jordan algebra and a, b, c, ε A. Then

$$\{a,\{b,\{a,c,a\},b\},a\} = \{\{a,b,a\},c,\{a,b,a\}\}$$
$$\{a,b,a\}\circ\{a,c,a\} = \{a,\{b,a^2,c\},a\} .$$

These formulae are of vital importance in establishing the properties of the inverse of an element of a Jordan algebra with unit.

DEFINITION. Let A be a Jordan algebra with unit, and let $a \in A$. Then a is invertible in A if there exists $b \in A$, called an inverse of A, such that

$$a \circ b = 1 \quad \text{and} \quad a^2 \circ b = a\ ,$$

Suppose a is invertible with an inverse b in a Jordan algebra A with unit. By the Shirshov-Cohn theorem, $Q(1,a,b)$ is special, and an easy calculation shows that $ab = ba = 1$, so that $Q(1,a,b)$ is an associative commutative algebra with unit. On the other hand if B is a Jordan subalgebra of an associative algebra with unit, such that $1 \varepsilon B$, and $a, b \varepsilon B$ are such that $ab = ba = 1$, then $a \circ b = 1$ and $a^2 \circ b = a$. So, there is no ambiguity about the use of the word invertible in either an associative subalgebra of a Jordan algebra or a Jordan subalgebra of an associative algebra. Further properties of inverses are given in the following Theorem, which is proved in [43] p52.

THEOREM 1·1·8. Let A be a Jordan algebra with unit, and let $a, b \varepsilon A$. Then (i) If a is invertible in A with an inverse b, then b is invertible in A with an inverse a.

(ii) The following three conditions are equivalent :

(a) a is invertible,

(b) $1 \varepsilon \operatorname{Im} U_a$,

(c) $(U_a)^{-1}$ exists.

(iii) If a is invertible, then it has a unique inverse $c = (U_a)^{-1} a$.

(iv) If a and b are inverses, $(U_b)^{-1} = U_a$ and $L_b = L_a (U_a)^{-1}$. Moreover $[L_{a^k}, L_{b^j}] = 0$ for all $k, j \in \mathbb{P}$. If we define $a^{-n} = b^n$ for $n > 0$, then $a^k \circ a^j = a^{k+j}$ for all integral k,j. Hence $Q(1,a,a^{-1})$ is an associative commutative algebra with unit.

(v) a and b are invertible if and only if $\{a,b,a\}$ is invertible.

As an application, we show the following spectral mapping Theorem.

LEMMA 1·1·9. Let A be a complex Jordan algebra with unit, $a \in A$, $\lambda \in C$ and p be a polynomial in one variable with complex coefficients. Then $p(a) - \lambda$ is not invertible in A if and only if $\lambda = p(\mu)$ for some $\mu \in \mathbb{C}$ such that $a - \mu$ is not invertible in A.

Proof. Let $q(z) = p(z) - \lambda = \prod_{j=1}^{n} (z - \mu_j)$. If $p(a) - \lambda$ is not invertible, then $q(a)$ is not invertible, and hence $U_{q(a)}$ and so $U_{(a-\mu_1)} \cdot U_{(a-\mu_2)} \cdots U_{(a-\mu_n)}$ is not invertible. Thus, for at least one j $(1 \leq j \leq n)$, $U_{a-\mu_j}$ and hence $a - \mu_j$ is not invertible. Moreover $0 = q(\mu_j) = p(\mu_j) - \lambda$, as required. The converse is proved by reversing the above argument.

For invertible elements, the Shirshov-Cohn theorem has been extended by McCrimmon [57], to give the following result.

THEOREM 1·1·10. Let A be a Jordan algebra with unit, and let a and b be invertible elements of A. Then $Q(1,a,a^{-1},b,b^{-1})$ is a special Jordan algebra with unit.

We conclude this section with some results about idempotents of a Jordan algebra.

DEFINITION. Let A be a Jordan algebra. An element $e \in A$ is called an idempotent if $e^2 = e$. If A is a complex algebra with involution, a self-adjoint idempotent is called a projection. If A is a real Jordan algebra, an idempotent in A is also called a projection.

LEMMA 1·1·11. Let A be a Jordan algebra, and let e be an idempotent in A. Let $A^j(e) = \{x \in A : e \circ x = j\,x\}$ for $j = 0,\frac{1}{2},1$. Then $A = A^1(e) \oplus A^{\frac{1}{2}}(e) \oplus A^0(e)$ as a linear space direct sum and

(i) $A^1(e) \circ A^1(e) \subseteq A^1(e)$, $A^0(e) \circ A^0(e) \subseteq A^0(e)$, $A^0(e) \circ A^1(e) = 0$,

(ii) $A^{\frac{1}{2}}(e) \circ A^{\frac{1}{2}}(e) \subseteq A^0(e) \oplus A^1(e)$, $A^{\frac{1}{2}}(e) \circ (A^0(e) \oplus A^1(e)) \subseteq A^{\frac{1}{2}}(e)$,

(iii) $[L_a, L_b] = 0$ if $a \in A^0(e)$ and $b \in A^1(e)$.

The proof of Lemma 1·1·11 may be found in [43] p. 118. The decomposition $A = A^1(e) \oplus A^{\frac{1}{2}}(e) \oplus A^0(e)$ is called the Pierce decomposition of A with respect to e .

DEFINITION Let A be a Jordan algebra, and $a,b \in A$. a and b operator commute if L_a commutes with L_b .

We note that Lemma 1·1·11 (iii) may be rephrased as a and b operator commute if $a \in A^0(e)$ and $b \in A^1(e)$. In a general Jordan algebra, it is not easy to determine whether two elements operator commute, but if one is an idempotent, the next Theorem gives a satisfactory characterisation. The proof is taken from [5] and [43].

THEOREM 1·1·12. Let A be a Jordan algebra, $a \in A$, and e an idempotent in A . Then the following are equivalent:

(i) a and e operator commute ,

(ii) $a \in A^0(e) \oplus A^1(e)$,

(iii) $L_e(a) = U_e(a)$.

If, in addition, A has a unit, then these three conditions are equivalent to

(iv) $a = U_e(a) + U_{1-e}(a)$.

<u>Proof</u> (i) $\Rightarrow$ (iii) As $(L_eL_a - L_aL_e)e = 0$, we have $L_eL_ae = L_ae$, so

$(L_e)^2 a = L_e a$. Hence

$$U_e(a) = 2(L_e)^2(a) - L_e a = L_e a \ .$$

(iii) $\Rightarrow$ (ii) Suppose $a = b + c + d$ where $b \in A^0(e)$, $c \in A^{\frac{1}{2}}(e)$ and $d \in A^1(e)$. Then

$$L_e(d) = d = 2eo(eod) - eod = U_e(d)$$

and

$$L_e(b) = 0 = 2eo(eob) - eob = U_e(b) \ .$$

Hence, by hypotheses $U_e(c) = L_e(c)$. However

$$\begin{aligned} \tfrac{1}{2}c = L_e(c) &= U_e(c) \\ &= 2eo(eoc) - eoc \\ &= 0 \end{aligned}$$

So $c = 0$, and thus $a \in A^0(e) \oplus A^1(e)$

(ii) $\Rightarrow$ (i) Let $a = d + b$ where $b \in A^0(e)$ and $d \in A^1(e)$. By Lemma 1·1·11 (iii), $[L_b, L_e] = 0$. If A has a unit, then it is clear that $d \in A^0(1-e)$, so $[L_d, L_{1-e}] = 0$, and thus $[L_d, L_e] = 0$. Hence $[L_a, L_e] = 0$. The general case follows from Lemma 1·1·3.

Now assume A has a unit.

(iii) $\Rightarrow$ (iv) As $L_e(x) = \frac{1}{2}(x + U_e(x) - U_{1-e}(x))$ for all x in A , if $L_e(a) = U_e(a)$, then $a = U_e(a) + U_{1-e}(a)$.

(iv) $\Rightarrow$ (iii) Conversely, if $a = U_e(a) + U_{1-e}(a)$,

$$L_e(a) = U_e(a) \ .$$

DEFINITION. Let A be a Jordan algebra, and let $e_1, e_2, \ldots, e_n$ be idempotents. Then $e_1, \ldots, e_n$ are orthogonal if $e_i \, o \, e_j = 0$ for $i \neq j$.

If A is a Jordan algebra with unit, and $e_1, \ldots, e_n$ are orthogonal idempotents in A such that $\sum_{j=1}^{n} e_j = 1$, there is a notion of a Pierce decomposition with respect to $e_1, \ldots, e_n$, [43], and this is used to get a deeper structure theory for Jordan algebras. Although we do not use this directly, it is used in the proof of the "Halving Lemma", in [5] and in the classification of finite dimensional semisimple real Jordan algebras in [43].

1·2 Banach Jordan algebras

DEFINITION. Let $(A, \wedge)$ be a non-associative algebra. We call A a Banach non-associative algebra if there is a norm $\|.\|$ on A such that $(A, \|.\|)$ is a Banach space and

$$\|a \wedge b\| \leq \|a\| \|b\|$$

for all a and b in A. If A has a unit, A is unital if $\|1\| = 1$.

We shall mainly be concerned with Banach algebras (we keep the convention that a Banach algebra is an associative algebra), and Banach Jordan algebras, although it is more convenient at times to consider arbitrary non-associative algebras.

Example 1·2·1. Let A be a Banach algebra. Then any closed Jordan subalgebra of A is a Banach Jordan algebra.

Example 1·2·2. Let $\mathcal{H}$ be a real Hilbert space of dimension at least three. Let e be any vector of norm one in $\mathcal{H}$, and let N be the orthogonal complement of $Sp(e)$. Then if $(.,.)$ denotes the inner product on $\mathcal{H}$, $\mathcal{H}$, with product

$$(\alpha e+a)o(\beta e+b) = (\alpha\beta+(a,b))e + (\alpha b+\beta a)$$

and norm

$$\|\alpha e+a\|^2 = |\alpha|^2 + (a,a)$$

where $\alpha,\beta \in \mathbb{R}$ and $a,b \in N$, is a real Banach Jordan algebra.

The latter example is given in Topping [85], and the Jordan algebra obtained is called a spin factor.

The purpose of this section is to develop the elementary theory of Banach Jordan algebras. We first introduce the following notation.

Notation Let A be a Banach Jordan algebra, and let $x_1,\ldots,x_r \in A$. We let $P(x_1,\ldots,x_r)$ denote the norm closure of $Q(x_1,\ldots,x_r)$.

LEMMA 1·2·1. Let A be a Banach Jordan algebra, and let $x_1,\ldots,x_r \in A$.

(i) Multiplication is jointly continuous in the norm topology.

(ii) $P(x_1,\ldots,x_r)$ is a closed subalgebra of A.

(iii) If $Q(x_1,\ldots,x_r)$ is an associative subalgebra of A, $P(x_1,\ldots,x_r)$ is a commutative Banach algebra.

Proof (i). This follows from the submultiplicative property of the norm, in the same way as [66] p228 equn.9.

(ii) and (iii) are easy consequences of (i).

Our first main topic is the spectrum of an element.

DEFINITION. Let A be a complex unital Banach Jordan algebra, and let $x \in A$. Let $\rho_A(x) = \{\lambda \in \mathbb{C} : x-\lambda \text{ is invertible in } A\}$, and let $\sigma_A(x) = \mathbb{C} \setminus \rho_A(x)$. $\sigma_A(x)$ is called the spectrum of x in A.

When no confusion can arise, we shall write $\rho(x)$ and $\sigma(x)$ in place of $\rho_A(x)$ and $\sigma_A(x)$. Most of the results we present on $\sigma(x)$

are found in [27] , in slightly greater generality. Our approach follows the pattern in [66] for Banach algebras.

LEMMA 1·2·2. Let A be a complex unital Banach Jordan algebra.

(i) If $x \in A$ and $\|x\|<1$, then $1-x$ is invertible, and $(1-x)^{-1} \in P(1,x)$.

(ii) Let x_n be a sequence of invertible elements of A converging to an invertible element x of A. If $h_n = x_n - x$, then, whenever $\|h_n\| \leq {}^1/_6\|x^{-1}\|^{-1}$,

$$\|x_n^{-1} - x^{-1} + \{x^{-1}, h_n, x^{-1}\}\| \leq 18\|h_n\|^2\|x^{-1}\|^3 ,$$

and in particular $x_n^{-1} \to x^{-1}$.

(iii) The set of invertible elements of A is open.

Proof. (i) By Theorem 1·1·5 and Lemma 1·2·1 , $P(1,x)$ is a commutative unital Banach algebra. The result now follows from the well-known result that $(1-x)^{-1} = \sum_{n=o}^{\infty} x^n$.

(ii) By McCrimmon's Theorem, $Q(1,x,x_n,x_n^{-1},x^{-1})$ is special. So, as

$$x^{-1}(x-x_n)x_n^{-1} = x_n^{-1}- x^{-1} = x_n^{-1}(x-x_n)x^{-1},$$

$$x_n^{-1} - x^{-1} = \{x^{-1},(x-x_n), x_n^{-1}\} .$$

Hence, if $\|h_n\| \leq \frac{1}{6}\|x^{-1}\|^{-1}$, as $\|U_a(b)\| \leq 3\|a\|^2\|b\|$ for all a and b in A,

$$\begin{aligned}\|x_n^{-1}\|-\|x^{-1}\| &\leq \|x_n^{-1} - x^{-1}\| \\ &\leq 3\|x^{-1}\|\|x - x_n\|\|x_n^{-1}\| \\ &\leq \tfrac{1}{2}\|x_n^{-1}\| .\end{aligned}$$

Thus $\|x_n^{-1}\| \leq 2\|x^{-1}\|$. Moreover,

$$\begin{aligned}\|x_n^{-1} - x^{-1} + \{x^{-1},h_n,x^{-1}\}\| &= \|\{x_n^{-1},h_n,x^{-1}\} - \{x^{-1},h_n,x^{-1}\}\| \\ &= \|\{x_n^{-1} - x^1 ,h_n,x^{-1}\}\| \\ &\leq 3\|x_n^{-1} - x^{-1}\| \|h_n\| \|x^{-1}\| \\ &\leq 9\|x_n^{-1}\| \|h_n\|^2\|x^{-1}\|^2 \\ &\leq 18\|h_n\|^2\|x^{-1}\|^3 ,\end{aligned}$$

In particular, as $h_n \to 0$, $x_n^{-1} \to x^{-1}$.

(iii) Suppose a is an invertible element of A . As $U_{a^{-1}}$ is a continuous linear operator, there is an open set N containing a^2 such that $U_{a^{-1}}(N) = \{x \in A : \|1-x\|<1\}$. As the map $f(x)=U_x(1)$ is continuous, there is an open set M , containing a , such that $f^{-1}(N) = M$. If $y \in M$, then $f(y) = y^2 \in N$, so $U_{a^{-1}} y^2$ is invertible. By Theorem 1·1·8, y^2 is invertible, and thus y is invertible.

THEOREM 1·2·3. Let A be a complex unital Banach Jordan algebra, and let $a \in A$. Then $\sigma(a)$ is a compact non-empty subset of $\mathbb{C}$.

Proof We first note that if $\lambda \in \mathbb{C}$ and $|\lambda| > \|a\|$, then $\|\lambda^{-1}a\| < 1$, and so $\lambda \in \rho(a)$. Hence $\sigma(a)$ is a bounded set in $\mathbb{C}$. $\sigma(a)$ is closed by Lemma 1·2·2, and so $\sigma(a)$ is compact.

Suppose $\sigma(a) = \emptyset$. By the Hahn-Banach Theorem, there exists $f \in A'$ such that $f(a^{-1}) \neq 0$. Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(\lambda) = f((\lambda-a)^{-1})$$

As $\sigma(a) = \emptyset$, g is a well defined function, and by Lemma 1·2·2 g is entire. Moreover, if $|\lambda| \geq 2\|a\|$,

$$\begin{aligned} \|(\lambda-a)^{-1}\| &= |\lambda|^{-1} \, \|\sum_{n=0}^{\infty}(\lambda^{-1}a)^n\| \\ &\leq |\lambda|^{-1} \sum_{n=0}^{\infty}|\lambda^{-1}| \, \|a^n\| \\ &= |\lambda|^{-1}(1-|\lambda^{-1}|\|a\|)^{-1} \\ &\leq 2|\lambda|^{-1} , \end{aligned}$$

so that g is bounded. Hence, by Liouville's Theorem, g is constant, and as $g(\lambda) \to 0$, as $|\lambda| \to \infty$, $g(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. This contradicts $f(a^{-1}) = g(0) \neq 0$, so $\sigma(a) \neq \emptyset$.

DEFINITION Let A be a complex unital Banach Jordan algebra, and let $a \in A$. The spectral radius of a, denoted by $r(a)$ is defined by

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$$

Before the next Theorem, we require a topological Lemma, whose proof may be found in [66].

LEMMA 1·2·4. Suppose V and W are open sets in a topological space X, such that $V \subseteq W$ and W contains no boundary points of V. Then V is a union of components of W.

THEOREM 1·2·5. Let A be a complex unital Banach Jordan algebra, and let J be a closed subalgebra of A containing the unit. If $a \in J$, then $\sigma_J(a)$ is the union of $\sigma_A(a)$ and a (possibly empty) collection of bounded components in the complement of $\sigma_A(a)$. In particular, the boundary of $\sigma_J(x)$ is contained in $\sigma_A(x)$, and

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}$$

Proof. For the first part, it suffices to show that $\rho_J(a)$ is the union of certain components of $\rho_A(x)$, and hence, by Lemma 1·2·4, that every boundary point of $\rho_J(a)$ is not in $\rho_A(a)$. Let λ be a boundary point of $\rho_J(a)$. As $\rho_J(a)$ is open, $\lambda \notin \rho_J(a)$, but there is a sequence $\{\lambda_n\}$ in $\rho_J(a)$ such that $\lambda_n \to \lambda$. Suppose $\lambda \in \rho_A(a)$. As inversion is continuous, $(a-\lambda_n)^{-1} \to (a-\lambda)^{-1}$, and in particular

$$\{\|(a-\lambda_n)^{-1}\| : n \in \mathbb{N}\} \leq K$$

for some $K \in \mathbb{R}^+$. Choose $n \in \mathbb{N}$ such that $|\lambda-\lambda_n| \leq (2K)^{-1}$. By Theorem 1·1·8 and Lemma 1·2·1, $E = P(1, a-\lambda_n, (a-\lambda_n)^{-1}) \subseteq J$ is a commutative Banach algebra, $a - \lambda_n$ is invertible in E, and

$$\|1 - (a-\lambda_n)^{-1}(a-\lambda)\| = \|(a-\lambda_n)^{-1}\| \, |\lambda-\lambda_n| < \tfrac{1}{2}.$$

Hence $a - \lambda$ is invertible in E, which is a contradiction.

To complete the proof, it remains to show that $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$. However, this is immediate from the same result in Banach algebras, and the first part above, if we take $J = P(1,a)$.

COROLLARY 1·2·6. Let A be a complex unital Banach Jordan algebra, and J be a closed subalgebra containing the unit. If $a \in J$ is such that $\sigma_A(x)$ is simply connected, that is $\mathbb{C}\setminus\sigma_A(x)$ is connected, then $\sigma_A(x) = \sigma_J(x)$.

Although we have developed the spectrum for complex unital Banach Jordan algebras, we now extend it to two other situations we are interested in. These are analogues of results in [14].

LEMMA 1·2·7. Let A be a Banach Jordan algebra over $\mathbb{F}$ without unit. Then $A \oplus \mathbb{F}$ with product and norm defined by

$$(a,\mu)\circ(b,\lambda) = (a\circ b+\mu b+\lambda a,\mu\lambda),$$

$$\|(a,\mu)\| = \|a\| + |\mu| ,$$

for a and b in A, and λ and μ in $\mathbb{F}$, is a unital Banach Jordan algebra.

THEOREM 1·2·8. Let A be a real Banach Jordan algebra, and let $A\oplus iA$ be its complexification. Then there is a norm p on $A\oplus iA$ such that

(i) $A\oplus iA$ is a complex Banach Jordan algebra,

(ii) $\max(\|a\|,\|b\|) \leq p(a+ib) \leq 2\max(\|a\|,\|b\|)$ for a and b in A,

(iii) $p(a,0) = \|a\|$ for a in A.

Proof. Let $S = \{a \in A:\|a\|<1\}$, and let $V = |co|(S\times\{0\})$. Clearly V is convex and balanced. Given $(a,b) \in A+iA$ and $\mu > \max(\|a\|,\|b\|)$, we have

$$\tfrac{1}{2}(\mu)^{-1}(a,b) = \tfrac{1}{2}(\mu^{-1}a,0) + \tfrac{1}{2}i(\mu^{-1}b,0) \in V ,$$

so that V is absorbent. Moreover, if p is the Minkowski functional of V, that is, $p(x) = \inf\{t > 0 : t^{-1}x \in V\}$, p is a semi-norm on $A\oplus iA$ such that

$$p((a,b)) \leqslant 2 \max(\|a\|,\|b\|) .$$

Conversely, if $(a,b) \in V$, so that $(a,b) = \sum_{k=1}^{n} (\beta_k+i\gamma_k)(u_k,0)$ where $\beta_k, \gamma_k \in \mathbb{R}$ $\sum_{k=1}^{n}|\beta_k+i\gamma_k| \leqslant 1$ and $u_k \in S$, then $a = \sum_{k=1}^{n}\beta_k u_k \in S$ and $b = \sum_{k=1}^{n} \gamma_k u_k \in S$, so that $\|a\|<1$ and $\|b\|<1$. Hence, we have

$$\max(\|a\|,\|b\|) \leqslant p(a,b) ,$$

and so p is a norm on $A\oplus iA$ under which $A\oplus iA$ is a Banach space. In particular $\|a\| \leqslant p(a,0)$ for a in A, so as $x \in S$ implies $(x,0) \in V$,

$$\|a\| = p(a,0)$$

for a in A.

To show that $(A\oplus iA,p)$ is a Banach Jordan algebra, it suffices to show that if (a,b) and (c,d) are in V, then so is $(a,b)\circ(c,d)$. Hence suppose $(a,b) = \sum_{k=1}^{n}\alpha_k(u_k,0)$ and $(c,d) = \sum_{j=1}^{m}\lambda_j(v_j,0)$ where $u_k, v_j \in S$, and $\sum_{k=1}^{n}|\alpha_k| \leqslant 1$ and $\sum_{j=1}^{m}|\lambda_j| \leqslant 1$. Then

$$(a,b)\circ(c,d) = \sum_{k=1}^{n}\sum_{j=1}^{m}\alpha_k\lambda_j\ (u_k\circ v_j,0) \in V$$

as $u_k\circ v_j \in S$ and $\sum_{k=1}^{n}\sum_{j=1}^{m}|\alpha_k\lambda_j| \leqslant 1$.

These results allow us to make the following definitions.

DEFINITION (i) Let A be a real unital Banach Jordan algebra. Given $a \in A$, we define $\sigma(a)$ as the spectrum of a in the complexification of A under the norm constructed in Theorem 1·2·8.

(ii) Let A be a Banach Jordan algebra without unit. Given $a \in A$, we define $\sigma(a)$ as the spectrum of a in $A \oplus F$ with norm and product defined in Lemma 1·2·7.

As for Banach algebras, the spectrum of an element is related to the characters on A, that is, the non-zero homomorphisms from A into $\mathbb{C}$. Although we shall not need Theorem 1·2·10 we state it for completeness, and remark that the proof is similar to the one for Banach algebras in [66].

LEMMA 1·2·9. Let A be a complex unital Banach Jordan algebra, I a proper ideal of A, and x an invertible element of A. Then

(i) $x \notin I$.

(ii) I is contained in a maximal ideal.

(iii) Maximal ideals of A are closed.

(iv) If I is closed, A/I, with the quotient norm, is a complex unital Banach Jordan algebra.

Proof (i) If $x \in I$, then $1 = x \circ x^{-1} \in I$, a contradiction.

(ii) This follows from Zorn's Lemma.

(iii) A maximal ideal M is proper, so as $T = \{x : \|1-x\|<1\}$ is an open set of invertible elements, $M \cap T = \emptyset$. Hence, as T is open, $\overline{M} \cap T = \emptyset$, and hence, by maximality, $M = \overline{M}$. So M is closed.

(iv) A/I is a complex Banach space under the quotient norm. Further, as $I \cap T = \emptyset$,

$$1 \geq \|1+I\| \geq 1,$$

so $\|1+I\| = 1$. Finally, as A is a Banach Jordan algebra, so is A/I.

THEOREM 1·2·10. Let A be a complex unital Banach Jordan algebra, and let Γ_A denote the set of all characters on A. Let $x \in A$ and $\psi \in \Gamma_A$.

(i) $\psi(1) = 1$, and if x is invertible, $\psi(x) \neq 0$.

(ii) $\psi|_{P(1,x)}$ is a character on $P(1,x)$.

(iii) $\psi(x) \in \sigma(x)$.

(iv) $\|\psi\| \leq 1$.

(v) Γ_A is a $\tau(A,A')$ compact subset of A'.

Finally, we note the following Gelfand-Mazur Theorem for Banach Jordan algebras. Again, we shall not require it in the sequel.

THEOREM 1·2·11. Let A be a complex unital Banach algebra in which every non-zero element is invertible. Then A is isometrically isomorphic to $\mathbb{C}$.

For the remainder of the section, we give some of the properties of a construction due to Arens, [7].

DEFINITION Let $(A, \wedge)$ be a Banach non-associative algebra, $a,b \in A$, $f \in A'$ and $\alpha, \beta \in A''$.

(i) Define $f_a \in A'$ by $f_a(b) = f(a \wedge b)$.

(ii) Define $\beta_f \in A'$ by $\beta_f(a) = \beta(f_a)$.

(iii) Define $\alpha \square \beta \in A''$ by $(\alpha \square \beta)(f) = \alpha(\beta_f)$.

The following Theorem is due to Arens [7].

THEOREM 1·2·12. Let $(A, \wedge)$ be a Banach non-associative algebra, and let $j : A \to A''$ be the natural embedding. Then $j(a \wedge b) = j(a) \square j(b)$, and $(A'', \square)$ is a Banach non-associative algebra. Moreover, if 1 is a unit for A, $j(1)$ is a unit for A'', and A'' is associative if A is. If $(B, \wedge)$ is another Banach non-associative algebra, and $T : A \to B$ a continuous homomorphism, then T'' is a continuous homomorphism of $(A'', \square)$ into $(B'', \square)$.

In order to avoid excess notation, if $(A, \wedge)$ is a Banach non-associative algebra, we shall also use the symbol $\wedge$ to denote the Arens product in A''. In view of Theorem 1·2·12, no confusion will arise.

Let $(A, \wedge)$ be a Banach non-associative algebra. A second product on A may be constructed by $a \vee b = b \wedge a$ for a and b in A. It is easy to see that $(A, \vee)$ with the original norm is a Banach non associative algebra, called the reversed algebra of A. A is called (Arens) regular if

$$a \vee b = b \wedge a$$

for all a and b in A''.

Even commutative associative algebras may fail to be regular. The following example, due to Civin and Yood [23], shows that A'' may fail to be a commutative algebra, even if A is.

Example Let $L^1(\mathbb{R})$ be the group algebra of $\mathbb{R}$, that is, the commutative Banach algebra of all Lebesgue integrable complex valued functions on $\mathbb{R}$ for which

$$\|f\| = \int_{\mathbb{R}} |f(t)|\, dt < \infty$$

with the product

$$(f * g)(s) = \int_{\mathbb{R}} f(t)g(s-t)dt \qquad (s \in \mathbb{R}).$$

Then $(L^1(R))''$ with Arens product is a non-commutative Banach algebra.

The following properties do hold for Arens products, however.

LEMMA 1·2·13. Let A and B be Banach Jordan algebras over $\mathbb{F}$, and let $A \oplus B$ be the Banach Jordan algebra with pointwise addition, scalar multiplication and product, and norm

$$\|x+y\| = \max(\|x\|, \|y\|).$$

Then $(A \oplus B)''$ is isometrically isomorphic to $A'' \oplus B''$.

LEMMA 1·2·14. Let A be an Arens regular Banach non-associative algebra, and let E be a closed subalgebra of A. Then E is Arens regular.

LEMMA 1·2·15. Let A be a Banach algebra, and let $(A,.)$ be the reversed algebra of A. If (A,o) is the Banach Jordan algebra where o is defined by

$$a \circ b = \tfrac{1}{2}(ab + ba) = \tfrac{1}{2}(ab + a.b) ,$$

then the Arens products obtained from the three algebras are related by

$$\alpha \circ \beta = \tfrac{1}{2}(\alpha\beta + \alpha.\beta)$$

for α and β in A''. In particular, if A is Arens regular, (A'',o) is a Jordan algebra.

We conclude the section with two algebras which are Arens regular. The first is well known [69], Theorem 1·17·2, while the second is essentially due to Shultz, [76].

THEOREM 1·2·16. Let A be a B^* - algebra, which, in its universal representation acts on the Hilbert space $\mathcal{H}$. Then A is Arens regular, and there is an isometrical *- isomorphism F of A'' onto the weak operator closure of A in $B(\mathcal{H})$ such that $a_\alpha \to a \ (\tau(A'',A'))$ if and only if $F(a_\alpha) \to F(a)$ in the weak operator topology.

THEOREM 1·2·17. Let M be a finite dimensional Banach Jordan algebra over $\mathbb{F}$, let I be an index set, and let $\Sigma(M_\alpha : \alpha \in I)$ denote the Banach Jordan algebra of bounded functions from I into M , with pointwise addition, scalar multiplication and product, and norm

$$\|\{m_\alpha\}\| = \sup\{\|m_\alpha\| : \alpha \in I\} .$$

Then $(\Sigma(M_\alpha : \alpha \in I))''$ is a Banach Jordan algebra.

Proof. If X is the Stone-Cech compactification of the discrete space I , then $\Sigma(M_\alpha : \alpha \in I)$ is isometrically isomorphic to the space $C(X,M)$ of all continuous functions from X into M. In order to calculate $(C(X,M))''$, we represent $C(X,M)$ as a tensor product. If E and F are Banach spaces, $E \otimes F$ will denote their algebraic tensor product. The completion of $E \otimes F$ in a norm α will be written $E \otimes_\alpha F$. Two important norms on $E \otimes F$ are λ , the least cross norm whose dual norm is a cross norm, and γ , the greatest cross norm [71].

As $\phi : C_{\mathbb{F}}(X) \times M \to C(X,M)$ defined by

$$\phi(f,m) = f(x)m$$

for x in X is a bilinear map, there is a unique linear map $\psi : C_{\mathbb{F}}(X) \otimes M \to C(X,M)$ such that

$$\psi\left(\sum_{j=1}^{n} f_j \otimes m_j\right) = \sum_{j=1}^{n} f_j(x) m_j$$

for x in X . As M is finite dimensional, ψ is one to one and onto. Moreover, if $f, f_1, \ldots, f_n \in C_{\mathbb{F}}(X)$, and $m, m_1, \ldots, m_n \in M$, we have

$$\|\psi(\sum_{j=1}^{n} f_j \otimes m_j)\| = \sup \|\sum_{j=1}^{n} f_j(x)m_j\| : x \in X\}$$

$$\leq \sup\{\sup |\sum_{j=1}^{n} f(x)g(m_j)| : g \in (M')_1\} : x \in X\}$$

$$\leq \sup\{\sup |\sum_{j=1}^{n} F(f_j)g(m_j)| : g \in (M')_1\} : F \in ((C(X))')_1\}$$

$$= \|\sum_{j=1}^{n} f_j \otimes m_j\|_\lambda \quad ,$$

and $\|\psi(f\otimes m)\| = \|f\|.\|m\|$.

Also, given $F \in (C(X))'$ and $G \in M'$, if $\|\sum_{j=1}^{n} f_j(x)m_j\| \leq 1$ for all x in X , we have

$$|\sum_{j=1}^{n} f_j(x)G(m_j)| \leq \|G\| \text{ for all } x \text{ in } X \quad ,$$

so $|F(\sum_{j=1}^{n} f_j\ G(m_j)| \leq \|F\|.\|G\|$, and thus

$$|\sum_{j=1}^{n} F(f_j)G(m_j)| \leq \|F\|.\|G\| \quad .$$

Hence, $\|\psi(.)\|$ is a cross norm on $E\otimes F$, at most λ and whose dual norm is also a cross norm. So, by [71] Theorem 2·1 , ψ is an isometry from $C_{\mathbb{F}}(X)\otimes M$ with the λ norm onto $C(X,M)$. In particular, $C_{\mathbb{F}}(X)\otimes M$ is complete in the λ norm, and so

$$C(X,M) = C_{\mathbb{F}}(X)\otimes M = C_{\mathbb{F}}(X) \otimes_\lambda M \quad .$$

This result is stated in [37] p90.

Next, if we define a product on $C_{\mathbb{F}}(X)\otimes M$ by

$$(\sum_{j=1}^{n} f_j \otimes m_j) o (\sum_{k=1}^{t} g_k \otimes q_k) = \sum_{j=1}^{n} \sum_{k=1}^{t} f_j g_k \otimes (m_j o q_k) \ ,$$

then $(C_{\mathbb{F}}(X)\otimes M, o)$ is a non-associative algebra, and ψ is an algebra homomorphism, so that $(C_{\mathbb{F}}(X)\otimes M, o)$ is a Banach Jordan algebra.

As M is finite dimensional, by [32] Corollary 5·1, $(C_{\mathbb{F}}(X)\otimes_\lambda M)'$ is isometrically isomorphic to $(C_{\mathbb{F}}(X))'\otimes_\gamma M$. Again, as M is finite dimensional, by [71], Theorem 2·5, $(C_{\mathbb{F}}(X))'\otimes_\gamma M$ is isometrically isomorphic to $(C_{\mathbb{F}}(X))''\otimes_\lambda M$. As $(C_{\mathbb{F}}(X))''$ is algebraically isometrically isomorphic to $C_{\mathbb{F}}(Y)$ for some compact Hausdorff space Y, it follows that $(C(X,M))''$ is isometrically isomorphic to $C(Y,M)$. To complete the proof we have to show that the Arens product on $(C_{\mathbb{F}}(X) \otimes M)''$ is the tensor product of the Arens product on each factor, and for this, it suffices to consider only simple tensors. Hence, let $f,g \in C_{\mathbb{F}}(X)$, $F \in (C_{\mathbb{F}}(X))'$, and $\alpha,\beta \in (C_{\mathbb{F}}(X))''$, and let $m, n \in M$, $N \in M'$, and $\nu,\mu \in M''$. Then

$$\begin{aligned} (F\otimes N)_{(f\otimes n)}(g\otimes m) &= F\otimes N(fg\otimes n\circ m) \\ &= F(fg) . N(n\circ m) \\ &= F_f g . N_n m \\ &= (F_f \otimes N_n)(g\otimes m) \quad . \end{aligned}$$

So

$$\begin{aligned} (\alpha\otimes\nu)_{(F\otimes N)}(f\otimes n) &= (\alpha\otimes\nu)(F\otimes N)_{(f\otimes n)} \\ &= (\alpha\otimes\nu)(F_f\otimes N_n) \\ &= (\alpha F_f)(\nu N_n) \\ &= (\alpha_F f)(\nu_N n) \\ &= (\alpha_F\otimes\nu_N)(f\otimes n) \quad . \end{aligned}$$

Thus

$$\begin{aligned} (\beta\otimes\mu)\circ(\alpha\otimes\nu)(F\otimes N) &= \beta\otimes\mu((\alpha\otimes\nu)_{(F\otimes N)}) \\ &= (\beta\otimes\mu)(\alpha_F\otimes\nu_N) \\ &= \beta\alpha_F\otimes\mu\nu_N \\ &= ((\beta\circ\alpha)F)((\mu\circ\nu)N) \\ &= ((\beta\circ\alpha)\otimes(\mu\circ\nu))F\otimes N \quad . \end{aligned}$$

Hence

$$(\beta\otimes\mu)\circ(\alpha\otimes\nu) = (\beta\circ\alpha)\otimes(\mu\circ\nu) \quad .$$

This completes the proof.

1.3 JB-algebras.

The definition and theory of JB-algebras presented in this section is entirely due to Alfsen, Shultz and Størmer [5],[76]. Despite the fact that we shall rely very heavily on their main results, it is not feasible to present here many of the proofs, but we nevertheless give an outline of their approach.

DEFINITION. Let A be a real Banach Jordan algebra. A is a JB-algebra if, for all a and b in A, we have

$$\|a\|^2 = \|a^2\| \leq \|a^2 + b^2\| \quad .$$

We remark that Alfsen, Shultz and Størmer only consider unital JB-algebras.

Example. If J is a JC-algebra, that is, a closed real Jordan subalgebra of self-adjoint operators of a C*-algebra, then J is a JB-algebra.

It might be expected that, conversely, every JB-algebra would be isometrically isomorphic to a JC-algebra. However, from work of Sherman [75], it is shown in [5] that there is a norm on M_3^8 under which it is a unital JB-algebra. So, not all JB-algebras are special, and this is one reason why the work involved in getting a suitable Gelfand-Neumark theorem for unital JB-algebras is somewhat long and tricky in places.

We now give some of the elementary theory of JB-algebras.

DEFINITION. Let A be a unital JB-algebra, and let $A^2 = \{a^2 : a \in A\}$.

We define an ordering on A by $a \geq b$ if, and only if, $a - b \in A^2$. The elements a such that $a \geq 0$ are called positive.

LEMMA 1·3·1. Let A be a unital JB-algebra.

(i) Given $x \in A$, the following are equivalent :

(a) $\|\alpha 1 - x\| \leq \alpha$ for all $\alpha \geq \|x\|$,

(b) $\|\alpha 1 - x\| \leq \alpha$ for some $\alpha \geq \|x\|$,

(c) $x = y^2$ for some $y \in P(1,x)$,

(d) $x \in A^2$.

(ii) A^2 is a proper closed convex cone.

(iii) A is formally real, that is, if $a_j \in A$ $(1 \leq j \leq n)$ and $\sum_{j=1}^{n} a_j^2 = 0$, then $a_j = 0$ $(1 \leq j \leq n)$.

Proof.(i) The only non-trivial steps are (b)⇒(c) and (d)⇒(a).
(b)⇒(c). Let $z = \alpha^{-1}x - 1$. Then $z \in A_1$, so, by the binomial series for square roots, there exists $w \in P(1,z) = P(1,x)$ with $w^2 = 1 + z$. If $y = \alpha^{\frac{1}{2}}w$, then $y^2 = x$.
(d)⇒(a). Suppose $x = y^2$, and let $\alpha \geq \|x\|$. If $z = -\alpha^{-1}x$, then $z \in A_1$, and so there exists $w \in P(1,z)$ such that $1 + z = w^2$. If $v = \alpha^{\frac{1}{2}}w$, then

$$-x + \alpha 1 = v^2 + w^2 - w^2 = v^2,$$

so

$$\|\alpha 1 - x\| = \|v^2\| \leq \|v^2 + y^2\| \leq \alpha .$$

(ii) By (i), $A^2 = \{a \in A : \| \|a\| - a\| \leq \|a\|\}$, and so A^2 is a closed convex cone. Further, if $x^2 = -y^2$, then

$$0 \leq \|x\|^2 = \|x^2\| \leq \|x^2+y^2\| = 0 ,$$

and thus $x = 0$. Hence $A^2 \cap -(A^2) = \{0\}$, so A is proper.
(iii) This follows from (ii).

It follows that $(A,\leq)$ is a partially ordered linear space. In fact, from [5], Theorem 2·1, $(A,\leq)$ is a complete order unit space, with the order unit norm coinciding with the given norm.

THEOREM 1·3·2. Let A be a unital JB-algebra, and M a closed associative subalgebra containing the unit. Then M is isometrically, order, and algebraically isomorphic to $C_{\mathbb{R}}(X)$, for some compact Hausdorff space X.

Proof. By Lemma 1·3·1 (i), if $a, b \in M$, and $a \geq 0$, $b \geq 0$, then there exist $c, d \in M$ such that $c^2 = a$, and $d^2 = b$. As M is associative, $a \circ b = (c \circ d)^2 \geq 0$. The result now follows by a theorem of Stone, see [45] Section 3.

COROLLARY 1·3·3. Let A and B be unital JB-algebras, and let $\phi : A \to B$ be a Jordan homomorphism such that $\phi(1) = 1$ and $\mathrm{Ker}\phi = \{0\}$. Then ϕ is an isometry.

Proof. Let $a \in A$. As $P(1,a)$ and $\phi(P(1,a))$ are associative subalgebras containing the unit, and the closure of an associative subalgebra is associative, it follows from Theorem 1·3·2, and [69] Corollary 1·2·6 that

$$\|\phi(a)\| = \|a\|$$

COROLLARY 1·3·4. Let A be a unital JB-algebra, and let $a \in A$.

(i) $\sigma(a) \subseteq \mathbb{R}$.

(ii) $\sigma_A(a) = \sigma_{P(1,a)}(a)$.

(iii) $a \geq 0$ if and only if $\sigma(a) \subseteq \mathbb{R}^+$.

Proof. (i) Suppose $\alpha+\beta i \in \sigma(a)$, where $\alpha, \beta \in R$, and $\beta \neq 0$. Let $b = \beta^{-1}(a-\alpha)$, so that $i \in \sigma(b)$. By Lemma 1·1·9, $-1 \in \sigma(b^2)$, so $-1 \in \sigma_{P(1,b)}(b^2)$. However, by Theorem 1·3·2, $P(1,b)$ is isometrically order and algebraically isomorphic to $C_{\mathbb{R}}(X)$, for some compact Hausdorff space X . This is a contradiction, as $-1 \notin \sigma_{C_{\mathbb{R}}(X)}(a^2)$ for any $a \in C_R(X)$.

(ii) This follows from (i), and Corollary 1·2·6.

(iii) This follows from (i) and (ii).

We remark that, in [5], the spectrum of an element of a unital JB-algebra is not based on the complexification. However, Corollary 1·3·4 (i) shows that the two definitions coincide.

It follows from Theorem 1·3·2, and Corollary 1·3·4, that if A is a unital JB-algebra, and $a \in A$, then $P(1,a)$ is isometrically, order, and algebraically isomorphic to $C_{\mathbb{R}}(\sigma(a))$. Hence, there is a well behaved continuous functional calculus for elements of A . In particular, suppose that a and b are positive, and $b^2 = a$. As $P(1,a) \subseteq P(1,b)$, and positive square roots are unique in a B^*-algebra, b is a norm limit of polynomials in a , and is the unique positive square root of a . Moreover, every $c \in A$ may be decomposed as $c = c^+ - c^-$, where $c^+ = (c^2)^{\frac{1}{2}}$, and $c^- = c^+ - c$ are positive.

THEOREM 1·3·5. Let A be a unital JB-algebra, and let $a \in A$. Then U_a is a positive linear operator, that is, if $b \in A^2$, then $U_a(b) \in A^2$.

We shall prove a more general result in Lemma 3·1·6, so, we omit the proof of Theorem 1·3·5. If A is a unital JB-algebra, and $a \in A$, as $a^2 \leq \|a^2\|$, it follows that

$$a^4 = U_a(a^2) \leq \|a^2\| \; U_a(1) = \|a^2\| a^2 \quad .$$

Thus, for every $b \in A^2$, we have

$$b^2 \leq \|b\|b .$$

The following Theorem is essentially proved in [5] Section 9. The proof is modeled on the analogous one for C*-algebras, as found in [28], and so is omitted here.

THEOREM 1·3·6. Let A be a unital JB-algebra, and let J be a closed subalgebra of A .

(i) J has an increasing approximate identity of positive elements in J_1 , that is, a net $\{e_\alpha\}$ in J such that $0 \leq e_\alpha \leq 1$ for all α , $e_\alpha \leq e_\beta$ whenever $\alpha \leq \beta$, and, for each a in J , we have

$$\lim \|a - e_\alpha \circ a\| = 0 .$$

(ii) If J is an ideal, then A/J is a unital JB-algebra.

We now turn to the promised brief outline of the methods and the main results in [5] . We start with definitions.

DEFINITION Let A be a unital JB-algebra.

(i) The state space of A is $\{f \in A' : f(1) = \|f\| = 1\}$.

(ii) A state f is called normal, if, whenever $\{a_\alpha\}$ is a decreasing net in A with infimum 0 , then $f(a_\alpha)$ is a decreasing net in $\mathbb{R}$ with limit 0.

(iii) A set S of states on A is full, if S is convex and if $a \geq 0$ if and only if $f(a) \geq 0$ for all $f \in S$.

(iv) A set S of states on A is called invariant if, for all b in A , and for all f in S , there exists $\lambda \in \mathbb{R}^+$ such that $fU_b \in \lambda S$.

(v) A is monotone complete if, whenever $\{b_\alpha\}$ is an increasing net in

A bounded above by an element of A , then there exists a supremum of $\{b_\alpha\}$ in A .

The first step in [5] is to embed an arbitrary unital JB-algebra A with state space K into its envelopping algebra $\tilde{A}$, which is a monotone complete unital JB-algebra for which K acts as a full invariant set of normal states. $\tilde{A}$ is a Jordan subalgebra of A" with Arens multiplication. The following result, [5] Lemma 4·2 and Propositions 4·3 and 4·9 , shows $\tilde{A}$ has many projections.

THEOREM 1·3·7. Let M be a monotone complete unital JB-algebra with a full invariant set of normal states K , and let $a \in A$. Then $W(1,a)$, the $\tau(M, Sp(K))$ closure of $P(1,a)$, is a monotone complete associative algebra isometrically isomorphic as an ordered algebra to a monotone complete $C_{\mathbb{R}}(X)$. Hence, there exists a unique set $\{e_\lambda : \lambda \in \mathbb{R}\}$ of idempotents in $W(1,a)$, called the spectral family of a, such that

(i) $e_\lambda \leq e_\mu$ if $\lambda \leq \mu$;

(ii) whenever $\{\mu_n\}$ is a monotone decreasing net in $\mathbb{R}$ with infimum λ , then e_{μ_n} is a monotone decreasing net in A with infimum e_λ ;

(iii) $e_\lambda = 0$ for $\lambda < -\|a\|$, and $e_\lambda = 1$ for $\lambda > \|a\|$;

(iv) $f(a^n) = \int \lambda^n \, d(f(e_\lambda))$, for all f in K ;

(v) the Stieltjes sums $\sum_{j=i}^{n} \lambda_{j-i}(e_{\lambda_j} - e_{\lambda_{j-1}})$ converge in norm to a as the mesh of the partition $\lambda_0 < \lambda_1 < \dots < \lambda_n$ of $[-\|a\| , \|a\|]$ tends to zero;

(vi) if $\mathcal{P}$ is the set of idempotents of M , then $\mathcal{P}$ is a complete orthomodular lattice.

In many ways, the von Neumann algebras which are easiest to handle are the factors. We shall see later that the set $Z(M)$ in the following definition is a reasonable choice for the centre of M .

DEFINITION. Let M be a monotone complete unital JB-algebra with a full invariant set of normal states. Let

$Z(M) = \{a \in M$: each element of the spectral family of a operator commutes with each element of $M\}$.

M is a JB-factor if $Z(M) = \mathbb{R}$.

THEOREM 1·3·8. Let A be a unital JB-algebra. Then for each $a \in A\backslash\{0\}$, there is a Jordan homomorphism $\phi_a : A \to \tilde{A}$ such that $\|\phi_a\| \leq 1$, $\phi_a(a) \neq 0$, and $\phi_a(A)$ is $\tau(\tilde{A},A')$ dense in a JB-factor.

This is [5] Proposition 5·6, and we now sketch a proof. Let K be the state space of A . By the Krein Milman theorem, there exists an extreme point μ of K such that $\mu(a) = \|a\|$. As the set of idempotents in $Z(\tilde{A})$ is a complete orthomodular lattice by [5] Lemma 5·1, there is a smallest idempotent p_μ in $Z(\tilde{A})$ such that $\mu(p_\mu) = 1$. If $\phi_a : A \to \tilde{A}$ is defined by $\phi_a(b) = U_{p_\mu}(b)$ for b in A , as p_μ is a central element of $\tilde{A}$, ϕ_a is a Jordan homomorphism, and, by choice of μ , $\phi_a(a) \neq 0$. Finally, as μ is extreme, $\mathrm{Im}\phi_a$ is a JB-factor, by [5] Lemma 5·5.

In view of [5] Proposition 5·6, Alfsen, Shultz and Størmer proceed to analyse JB-factors. We give their results after the next definition.

DEFINITION. Let M be a JB-factor.

(i) A non-zero idempotent e in M is minimal if, whenever $e = f + g$ where f and g are idempotents, either f or g is zero.

(ii) M is type I if M contains a minimal idempotent.

(iii) M is type I_n if n is the least upper bound of orthogonal idempotents, where $1 \leq n \leq \infty$.

THEOREM 1·3·9. Let M be a JB-factor. If M is type I_2, then M is a spin factor. If M is not of type I_2 , M is isomorphic to a Jordan matrix algebra $S_m(Q)$. If M is type I_n where $3 \leq n < \infty$, Q, and hence M, is finite dimensional, while if M is type I_n where $4 \leq n \leq \infty$, Q is associative.

In the next section, we shall sketch a proof, due to Harris [40], which shows that all spin factors are JC-algebras. By a modification of the Gelfand Neumark theorem for B^*-algebras, if Q is an associative algebra and $M = S_m(Q)$ is a JB-factor, then M is isometrically isomorphic to a JC-algebra, by [5] Theorem 8·6. If M is a JB-factor of type I_3 , then, by [5] Proposition 8·3, and the above remarks, either $M = M_3^8$ or M is isometrically isomorphic to a JC-algebra. Hence, we have the following Gelfand-Neumark theorem for JB-algebras, and two Corollaries.

THEOREM 1·3·10. Let A be a unital JB-algebra. There is a family G of Jordan homomorphisms $A \to \mathrm{Img}$ of norm one such that

(i) for all g in G , either Img is a JC-algebra or $\mathrm{Img} = M_3^8$;

(ii) for all a in $A \setminus \{0\}$, there exists $g \in G$ such that $g(a) \neq 0$.

COROLLARY 1·3·11. Let A be a unital JB-algebra, and let f be any s-identity for M_3^8 . Then the following are equivalent:

(i) A is special,

(ii) $f(a,b,c) = 0$ for all a,b and c in A ,

(iii) A is isometrically isomorphic to a JC-algebra.

COROLLARY 1·3·12. Let A be a unital JB-algebra, and let $a, b \in A$. Then $P(1,a,b)$ is a special unital JB-algebra.

Proof. If f is any s-identity for M_3^8 , then $f(x,y,z) = 0$ for all x, y and z in $Q(1,a,b)$ by the Shirshov-Cohn theorem. Hence, by the continuity of multiplication, $f(x,y,z) = 0$ for all x, y and z in $P(1,a,b)$. The result follows from Corollary 1·3·11.

We conclude this section with a restatement of Theorem 1·3·10 , and an application due to Shultz [76]. As we shall prove a similar result in Theorem 3.3.4, we shall not give the proof of the following Corollaries here.

COROLLARY 1·3·13. Let A be a unital JB-algebra. Then A is isometrically isomorphic to a real Jordan subalgebra of $C(X,M_3^8) \oplus S_1(B(\mathcal{H}))$, where X is a compact Hausdorff space, and $\mathcal{H}$ is a complex Hilbert space.

COROLLARY 1·3·14. If A is a unital JB-algebra, then A'', with Arens multiplication, is also a unital JB-algebra.

1.4 Holomorphic maps, and bounded symmetric homogeneous domains.

Apart from some well known definitions and examples, most of the work in this section is due to Harris [39],[40].

DEFINITION. Let X and Y be complex Banach spaces, and let V be an open subset of X.

(i) A map $f : V \to Y$ is called holomorphic if, for all x in V, there exists a unique element $Df(x)$ of $B(X,Y)$, called the Fréchet derivative of f at x, such that, as $h \to 0$,

$$\|h\|^{-1}\|f(x+h)-f(x)-(Df(x))(h)\| \to 0 .$$

(ii) A map $f : V \to V$ is called biholomorphic if f maps V one to one onto itself and both f and f^{-1} are holomorphic.

DEFINITION. Let X be a complex Banach space, and let V be a bounded open subset of X. V is a bounded symmetric homogeneous domain if, for all a in V there exists a biholomorphic map s of V onto itself such that s^2 is the identity, $s(a) = a$, and $s(z) \neq z$ if $z \neq a$, and for each b in V, there exists a biholomorphic map f_b of V onto itself such that $f_b(a) = b$.

Example 1·4·1. Let X and Y be complex Banach spaces, and let V be an open subset of X. If $f \in B(X,Y)$, then $f|_V$ is a holomorphic map, and $(Df|_V)(x) = f$.

Example 1·4·2. Let A be a complex unital Banach Jordan algebra, and let V be the open set of invertible elements of A. The map $f : V \to V$ defined by $f(x) = x^{-1}$ is a biholomorphic map of V onto V such that $Df(x) = -U_{x^{-1}}$.

Many results about holomorphic maps on $\mathbb{C}$ are also valid for complex Banach spaces [42], [60].

THEOREM 1·4·1. Let X, Y and Z be complex Banach spaces, and let $V \subseteq X$ and $W \subseteq Y$ be open sets. Let $f : V \to Y$, and $g : W \to Z$ be holomorphic maps, and suppose that $f(V) \subseteq W$. Then gf is a holomorphic map, and $Dgf(x) = Dg(f(x)).Df(x)$

COROLLARY 1·4·2. Let X be a complex Banach space, and let V be an open subset of X. If f is a biholomorphic map of V, and I is the identity map on X, $I(x) = Df^{-1}(f(x)).Df(x)$, so that $Df(x)$ is invertible, for all x in V, and $(Df(x))^{-1} = Df^{-1}(f(x))$.

One important result on holomorphic maps from $\mathbb{C}$ to $\mathbb{C}$ is the local existence and convergence of a power series expansion. A similar theorem is also true for holomorphic maps between complex Banach spaces, [60] p17, and this is used in [53] to prove the following extension of Cartan's uniqueness theorem. A more elementary proof is given in [39].

THEOREM 1·4·3. Let X be a complex Banach space and let V be its open unit ball. Suppose that $f : V \to V$ is a biholomorphic map such that $Df(0)$ is an isometry of X onto itself. Then f is linear, that if, $f(z) = Df(0)(z)$ for all z in V.

COROLLARY 1·4·4. Let X be a complex Banach space, and let V be its open unit ball. Suppose that $f : V \to V$ is a biholomorphic map such that $f(0) = 0$. Then f is the restriction of a linear isometry of X onto itself.

<u>Proof</u>. Let $x \in V$, and $g \in (X')_1$. Define $k : \Delta \to X$ by $k(z) = zx$ for z in Δ, and let $\phi(z) = gfk(z)$. Then ϕ is a holomorphic map of Δ into Δ with $\phi(0) = 0$. By Cauchy's inequalities, for all $\gamma \in \mathbb{R}$ such that $0 < \gamma < 1$

$$|D\phi(0)| \leq \gamma^{-1} \sup\{|\phi(z)| : |z| = \gamma\} \leq \gamma^{-1} .$$

Hence $|D\phi(0)| \leq 1$, and so, by the chain rule,

$$|g\, Df(0)k| \leq 1 .$$

Thus $|g\, Df(0)\, x| \leq 1$, and so, by the Hahn-Banach theorem, $\|Df(0)\| \leq 1$. Similarly $\|Df^{-1}(0)\| \leq 1$, and so, by Corollary 1·4·2, $Df(0)$ is an isometry of X onto itself. The result follows by Theorem 1·4·3.

The remainder of Harris' results we require are on what he called "J*-algebras". These are not algebras as we have defined them, so, we call them C^*-triple systems. Our applications of Harris' results will mainly be to JC^*-algebras.

DEFINITION. (Harris). Let $\mathcal{H}$ be a complex Hilbert space. A closed subspace E of $B(\mathcal{H})$ is called a C^*-triple system if, for all x in E, $xx^*x \in E$.

DEFINITION. Let $\mathcal{H}$ be a complex Hilbert space. A closed self-adjoint Jordan subalgebra of $B(\mathcal{H})$ is called a JC^*-algebra.

LEMMA 1·4·5. Let A be a C^*-triple system, and let a, b and $c \in A$. If p is any polynomial, then $p(ab^*)c + cp(b^*a)$ and $p(ab^*)cp(b^*a) \in A$.

THEOREM 1·4·6. Let A be a C^*-triple system with open unit ball V. For each $b \in V$, the Möbius transformation

$$T_b(a) = (1-bb^*)^{-\frac{1}{2}}(a+b)(1+b^*a)^{-1}(1-b^*b)^{\frac{1}{2}}$$

is a biholomorphic map of V onto V with $T_b(0) = b$. Moreover $(T_b)^{-1} = T_{-b}$, $(T_b(a))^* = T_{b^*}(a^*)$ and

$$(DT_b(a))(c) = (1-bb^*)^{\frac{1}{2}}(1+ab^*)^{-1}\, c(1+b^*a)^{-1}(1-b^*b)^{\frac{1}{2}}$$

for $a \in V$ and $c \in A$.

Proof. $(1-bb^*)^{\frac{1}{2}}b = b(1-b^*b)^{\frac{1}{2}}$ by comparison of the power series expansions. Hence

$$T_b(a) = b + (1-bb^*)^{\frac{1}{2}}a(1+b^*a)^{-1}(1-b^*b)^{\frac{1}{2}} . \qquad (\dagger)$$

So, by Lemma 1·4·5, $T_b(a) \in A$. Differentiation of $(\dagger)$ shows that T_b has the derivative asserted, so T_b is holomorphic in V. Also, it is clear from $(\dagger)$, that $T_b(0) = b$, and $(T_b(a))^* = T_{b^*}(a^*)$.

By an argument essentially due to Potapov, [64] Chapter 1, Section 1,

$$1 - (T_b(a))^*T_b(a) = (1-b^*b)^{\frac{1}{2}}(1+a^*b)^{-1}(1-a^*a)(1+b^*a)^{-1}(1-b^*b)^{\frac{1}{2}} ,$$

so $1 - (T_b(a))^*T_b(a)$ is positive and invertible. Hence $T_b(a) \in V$. To complete the proof, it suffices to show that $S = T_bT_{-b}$ is the identity on V. However $S(0) = 0$, and $DS(0) = DT_b(-b)DT_{-b}(0) = I$, so, by Theorem 1·4·3, S is the identity on V, as required.

COROLLARY 1·4·7. Let A be a C^*-triple system with open unit ball V. Every biholomorphic map $f : V \to V$ is of the form $f = ST_{-f^{-1}(0)}$, where $S : A \to A$ is a surjective linear isometry.

Proof. By Theorem 1·4·6, $S = f(T_{-f^{-1}(0)})^{-1}$ is a biholomorphic map of V onto itself, with $S(0) = 0$. Hence, by Corollary 1·4·4, S is a linear isometry.

We note that Theorem 1·4·6 shows that the open unit ball of a C^*-triple system is a bounded symmetric homogeneous domain. A different application is given in [14] to derive a strong version of the Russo-Dye theorem for unital C^*-algebras [67]. In fact, the proofs also work for unital JC^*-algebras, and so we omit the details.

LEMMA 1·4·8. Let A be a unital JC^*-algebra, and let $x \in A$ be such that $0 < \|x\| < 1$. Define $F : \{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|^{-1}\} \to A$ by

$$F(\lambda) = (1-xx^*)^{-\frac{1}{2}}(\lambda+x)(1+\lambda x^*)(1-x^*x)^{\frac{1}{2}} .$$

Then F is a holomorphic map of $\|x\|^{-1}\Delta$ into A such that $F(\lambda)$ is unitary for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Hence x is in the closed convex hull of the unitary elements of A.

THEOREM 1·4·9. Let A be a unital JC^*-algebra. Then

$$A_1 = \overline{\mathrm{co}}\{\exp ih : h = h^* \text{ and } h \in A\} .$$

Theorem 1·4·6 is used in [40] to derive a characterisation of the extreme points of the unit ball of a C^*-triple system. It generalises a result of Kadison [46].

THEOREM 1·4·10. Let A be a C^*-triple system, and let $a \in A_1$. Then, for all b in A with $\|b\| < \frac{1}{2}$, we have

$$\|a + (1-aa^*)^{\frac{1}{2}}b(1-a^*a)^{\frac{1}{2}}\| \leq 1 .$$

Hence a is an extreme point of A_1 if and only if

$$(1-aa^*)b(1-a^*a) = 0$$

for all b in A. Thus every extreme point of A_1 is a partial isometry.

We conclude this section with the promised sketch of Harris' proof that all spin factors are isometrically isomorphic to unital JC-algebras. We omit many of the algebraic proofs concerning the exterior algebra;

these are found in [35].

Let E be a real inner product space. For any $n \in \mathbb{N}$ we let $\otimes^n E$ denote the tensor product $E \otimes \ldots \otimes E$ (n times). If S_n denotes the group of all permutations on the set $\{1,2,\ldots,n\}$, each $\mu \in S_n$ induces an automorphism (also denoted by μ) of $\otimes^n E$ such that

$$\mu^{-1}(e_1 \otimes \ldots \otimes e_n) = e_{\mu(1)} \otimes \cdots \otimes e_{\mu(n)} .$$

If $\operatorname{sgn}\mu$ denotes the sign of the permutation μ, we define a linear map $\Pi_n : \otimes^n E \to \otimes^n E$ by

$$\Pi_n = (n!)^{-1} \sum_{\mu \in S_n} (\operatorname{sgn}\mu)\mu .$$

Let $\Lambda^n(E) = \otimes^n E/\operatorname{Ker}\Pi_n$. Then, if $\Lambda^0(E) = \mathbb{R}$, the exterior algebra, $\Lambda(E) = \oplus\{\Lambda^n(E): n \in \mathbb{P}\}$, consists of all sequences $\{z_n\}$ such that $z_n \in \Lambda^n(E)$ and only finitely many z_j are non-zero. We summarise the properties of $\Lambda(E)$ we require in the following Theorem.

THEOREM 1·4·11. Let E be a real inner product space, and let $\Lambda(E)$ be the exterior algebra of E. Then $\Lambda(E)$ is a real inner product space and an associative algebra under a product $\wedge$ such that

$$\Pi_p(x_1 \otimes \ldots \otimes x_p) \wedge \Pi_q(y_1 \otimes \ldots \otimes y_q) = \Pi_{p+q}(x_1 \otimes \ldots \otimes x_p \otimes y_1 \otimes \ldots \otimes y_q),$$

$1 \in \Lambda^0(E)$ is a unit for $\Lambda(E)$, and $y \wedge y = 0$ for $y \in \Lambda^1(E)$.

Further, for all x and z in $\Lambda(E)$, and $y \in \Lambda^1(E)$, we have

$$\langle L_y z, x\rangle = \langle z, \delta_y x\rangle$$

where δ_y is a linear operator on $\Lambda(E)$ such that $\delta_y(1) = 0$, $\delta_y(p) = (y,p)$ for p in $\Lambda^1(E)$ and

$$\delta_y(u \wedge v) = (\delta_y u) \wedge v + (-1)^p u \wedge \delta_y v$$

for $u \in \Lambda^p(E)$ and $v \in \Lambda(E)$.

THEOREM 1·4·12. Let E be a real Hilbert space with dimension at least three, let e be a vector of norm one, and let (E,o) be the Jordan algebra with unit e defined in Example 1·2·2. Let $\mathcal{H}$ be the completion of the real inner product space $\Lambda(E)$. Then, there is a continuous Jordan isomorphism of E into $B(\mathcal{H})$.

<u>Proof.</u> Given $x \in E$, let $A_x = L_x + \delta_x$. Then A_x is a linear operator on $\Lambda(E)$, and, for y in $\Lambda(E)$, we have

$$\begin{aligned}(A_x)^2(y) &= (L_x{}^2 + \delta_x{}^2 + L_x\delta_x + \delta_x L_x)(y) \\ &= 0 + 0 + L_x\delta_x y + \delta_x(x \wedge y) \\ &= x \wedge (\delta_x y) + (\delta_x x) \wedge y - x \wedge (\delta_x y) \\ &= (x,x)y \quad .\end{aligned}$$

Thus, $A_x{}^2 = (x,x)I$, and so

$$\|A_x y\|^2 = (A_x y, A_x y) = (A_x{}^2 y, y) = (x,x)\|y\|^2.$$

Hence, A_x is a continuous linear operator on $\Lambda(E)$, and so may be extended to a unique element of $B(\mathcal{H})$, also denoted by A_x.

Now define $\lambda : E \to B(\mathcal{H})$ by $\lambda(\alpha e+x) = \alpha + A_x$ for α in $\mathbb{R}$ and x in $(Sp(e))^{\perp}$. Then

$$\begin{aligned}\lambda((\alpha e+x)^2) &= \lambda((\alpha^2+(x,x))e + 2\alpha x) \\ &= \alpha^2 + (x,x) + 2\alpha A_x \\ &= \alpha^2 + A_x{}^2 + 2\alpha A_x \\ &= (\alpha+A_x)^2 \quad ,\end{aligned}$$

and so λ is a Jordan homomorphism. Finally, as $\{\alpha_n e + a_n\}$ converges in the spin factor norm if and only if α_n and a_n converge, it follows that λ is a one to one continuous map.

Finally, we note that the injection of $\mathrm{Im}\lambda$ into a JC-algebra proceeds as in the proof of [5] Lemma 8·5.

CHAPTER 2

The Vidav Palmer theorem [86], [62], [15] characterises unital B^*-algebras in terms of norm and algebra structure alone. In this chapter, we introduce JB^*-algebras, and one of our main results is an analogue of the Vidav Palmer theorem for unital Banach Jordan algebras. To prove this, it is necessary to generalise the theory of the numerical range to Banach non-associative algebras, and this is what we do in the first two sections. In addition, if A is a complex unital Banach Jordan algebra, and $\mathrm{Her}A$ denotes the Hermitian elements of A, we show that if $\mathrm{Her}A$ is a Jordan algebra, then $\mathrm{Her}A$ is a JB-algebra. So, we can use the results of [5] to derive results about A.

In the third section, we give some of the "elementary" theory of unital JB^*-algebras, which we require to prove the Vidav Palmer theorem. The final section is mainly devoted to applications of the Vidav Palmer theorem. These include analogues of results in [15], for example, that a closed Jordan ideal of a unital JB^*-algebra is self-adjoint, and the quotient algebra is again a unital JB^*-algebra, and a dual characterisation of unital JB^*-algebras, essentially due to Moore ([59],[16]). We also give results which have no natural analogue in associative algebras, for example, that the complexification of a unital JB-algebra may be given a norm under which it is a unital JB^*-algebra. We deduce from this a Gelfand Neumark theorem for unital JB^*-algebras.

Many of these results have been submitted for publication ([93],[94],[96],[97]).

2·1 The Numerical Range in Banach non-associative algebras

Although we shall later seldom require such generality, throughout this section, we let $(A, \wedge)$ be a complex unital Banach non-associative algebra. We develop the theory of the numerical range of an element of A in this section, and, as the numerical range is essentially a linear concept, many results in [15] and [16] are also valid for A.

DEFINITION. The set of states of A, denoted by $D(A,1)$ is defined by

$$D(A,1) = \{f \in A' : f(1) = 1 = \|f\|\} .$$

Given $a \in A$, the numerical range of a, denoted by $V(A,a)$ is defined by

$$V(A,a) = \{f(a) : f \in D(A,1)\} ,$$

and the numerical radius of a, denoted by $v(A,a)$ is defined by

$$v(A,a) = \sup\{|\lambda| : \lambda \in V(A,a)\} .$$

LEMMA 2·1·1. $D(A,1)$ is a non-empty $\tau(A',A)$ compact convex subset of A'.

Proof. By the Hahn-Banach theorem, $D(A,1)$ is non-empty. Clearly $D(A,1)$ is a $\tau(A',A)$ closed convex subset of $(A')_1$. Hence, by the Banach Alaoglu theorem, $D(A,1)$ is $\tau(A',A)$ compact.

LEMMA 2·1·2. For all a and b in A, and all α and β in $\mathbb{C}$,

(i) $V(A,a)$ is a non-empty compact convex subset of $\mathbb{C}$;

(ii) $V(A,\alpha+\beta a) = \alpha + \beta V(A,a)$;

(iii) $V(A,a+b) \subseteq V(A,a) + V(B,b)$;

(iv) $v(A,a) \leq \|a\|$.

Proof (i) follows from Lemma 2·1·1, while the remaining three parts follow easily from the definitions.

LEMMA 2·1·3. Let B be a closed subalgebra of A containing the unit. Then, for all b in B,

$$V(B,b) = V(A,b) .$$

Proof By the Hahn-Banach theorem, the restriction map $f \to f|_B$ maps $D(A,1)$ onto $D(B,1)$. Hence

$$V(B,b) = V(A,b) .$$

In view of Lemma 2·1·3, when no confusion can arise, we shall write $D(1)$, $V(a)$ and $v(a)$ for $D(A,1)$, $V(A,a)$ and $v(A,a)$ respectively.

If $P(a,1)$ is a Banach algebra, we may derive many of the properties of $V(a)$ from the known properties of $V(P(a,1),a)$. We shall do this in the next section, while here we derive the properties of $V(A,a)$ from those of $V(B(A),L_a)$.

DEFINITION. Let X be a complex Banach space, and let $T \in B(X)$. The spatial numerical range of T is defined by

$$W(T) = \{f(Tx) : x \in X_1,\ f \in (X')_1 \text{ and } f(x) = 1\} .$$

THEOREM 2·1·4. If $a \in A$, then $V(A,a) = V(B(A),L_a) = W(L_a)$.

Proof Given $f \in D(A,1)$, $\|f\| = \|1\| = f(1) = 1$, and so, for all a in A, we have

$$f(a) = f(a \wedge 1) \in W(L_a) .$$

Hence, for all a in A, $V(A,a) \subseteq W(L_a)$.

Conversely, suppose $f \in A'$ and $x \in A$ such that $\|f\| = \|x\| = f(x) = 1$. Define $F \in A'$ by

$$F(b) = f(b \wedge x)$$

for b in A . Clearly $F(1) = 1$ and $\|F\| \leq \|f\|.\|x\| = 1$, so, for all a in A , we have $W(L_a) \subseteq V(A,a)$.

Hence $V(A,a) = W(L_a)$, and so, by Lemma 2·1·2 and [15] Theorem 9·4,

$$W(L_a) = V(A,a) = \overline{co}V(A,a) = \overline{co}W(L_a) = V(B(A),L_a) .$$

COROLLARY 2·1·5. If $a \in A$, then $v(a) \leq \|a\| \leq ev(a)$.

Proof. By [15] Theorem 4·1, $v(L_a) \leq \|L_a\| \leq ev(L_a)$.

As A is a unital Banach non-associative algebra, $\|a\| = \|L_a\|$, and hence the result follows from Theorem 2·1·4.

COROLLARY 2·1·6. Given $f \in A'$, there exists $\alpha_k \in \mathbb{R}^+$, $f_k \in D(1)$ $(1 \leq k \leq 4)$ such that $f = \alpha_1 f_1 - \alpha_2 f_2 + i(\alpha_3 f_3 - \alpha_4 f_4)$ and $\sum_{k=1}^{4} \alpha_k \leq \sqrt{2}\, e\|f\|$. Hence $A' = Sp(D(1))$.

Proof. Let $\Gamma = \{1, -1, i, -i\}$, and note that $\overline{\Delta} \subseteq \sqrt{2}\, co\Gamma$. Let $T = \{(\alpha_1,\alpha_2,\alpha_3,\alpha_4) : \alpha_k \in \mathbb{R}^+ \ (1 \leq k \leq 4)$, and $\sum_{k=1}^{4} \alpha_k = 1\}$. As $D(1)$ is compact in the $\tau(A',A)$ topology, the set

$$D(1) \times (-1)D(1) \times iD(1) \times (-i)D(1) \times T$$

is compact in the product topology, and $co(\Gamma D(1))$ is the image of this set under the map

$$(f_1,-f_2,if_3,-if_4,(\alpha_1,\alpha_2,\alpha_3,\alpha_4)) \rightarrow \alpha_1 f_1 - \alpha_2 f_2 + i\alpha_3 f_3 - i\alpha_4 f_4 .$$

Thus $co(\Gamma D(1))$ is a $\tau(A',A)$ closed convex subset of A' .

Let $f \in A'$, and suppose $\|f\| \leq (\sqrt{2}e)^{-1}$. If $f \notin co(\Gamma D(1))$, by the Hahn-Banach theorem, there exists $a \in A$ and $\eta \in \mathbb{R}^+ \setminus \{0\}$ such that

$$\sup\{Re g(a) : g \in co(\Gamma D(1))\} \leq Re f(a) - \eta .$$

Given $g \in D(1)$, choose $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $|g(a)| = \mathrm{Re}\, \lambda g(a)$. Then $\lambda \in \sqrt{2}\mathrm{co}\Gamma$, so there exists $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in T$ such that $\lambda = \sqrt{2}(\alpha_1-\alpha_2+ i\alpha_3-i\alpha_4)$. So $\lambda g \in \sqrt{2}\mathrm{co}(\Gamma D(1))$, and hence

$$|g(a)| = \mathrm{Re}\, \lambda g(a) \leq \sqrt{2}(\mathrm{Re} f(a) - \eta) \ .$$

As this holds for all $g \in D(1)$, we have

$$\|a\|.\|f\| \geq \mathrm{Re}\, f(a) > (\sqrt{2})^{-1}\, v(a) \geq (\sqrt{2}e)^{-1}\|a\| \geq \|f\|.\|a\| \ ,$$

which is a contradiction.

The above proof is essentially due to Sinclair, see [16], Lemma 31·2. We now turn to study the Hermitian elements of A.

DEFINITION (i) An element $a \in A$ is Hermitian if $V(a) \subseteq \mathbb{R}$. The set of Hermitian elements is denoted by HerA.

(ii) An element $f \in A'$ is a Hermitian functional if f is in the real linear span of $D(A,1)$. The set of Hermitian functionals is denoted by $H(A')$.

THEOREM 2·1·7 HerA is a closed real linear subspace of A, $\mathrm{HerA} \cap i\mathrm{HerA} = \{0\}$, and $\mathrm{HerA} \oplus i\mathrm{HerA}$ is closed. Moreover, the natural linear involution on $\mathrm{HerA} \oplus i\mathrm{HerA}$ is continuous.

Proof. Let $\{h_n\}$ be a sequence of elements in HerA, and suppose that $h_n \to h$, where $h \in A$. As, for all $f \in D(1), f(h_n) \to f(h)$, we have $f(h) \in \mathbb{R}$. Hence HerA is closed. By Lemma 2·1·2, HerA is a real linear subspace.

If $b \in \mathrm{HerA} \cap i\mathrm{HerA}$, for all $f \in D(1)$, we have $f(b) \in \mathbb{R} \cap i\mathbb{R} = \{0\}$. Thus $V(b) = 0$, and so, by Corollary 2·1·5, $b = 0$.

Next, suppose $\{h_n + ik_n\}$ is a sequence in A such that $h_n, k_n \in \mathrm{HerA}$

for all n, and $h_n + ik_n \to x$. For each m in $\mathbb{N}$, we have

$$e^{-1}\|h_m\| \leq v(h_m) \leq v(h_m + ik_m) \leq \|h_m + ik_m\| ,$$

and similarly $e^{-1}\|k_m\| \leq \|h_m + ik_m\|$. Thus $\{h_m\}$ and $\{k_m\}$ are Cauchy sequences in HerA, and so converge to elements h and k in HerA. As $x = \lim(h_n + ik_n) = h + ik$, we have $x \in \mathrm{HerA} \oplus i\mathrm{HerA}$, and so $\mathrm{HerA} \oplus i\mathrm{HerA}$ is closed. Moreover

$$x^* = h - ik = \lim(h_n + ik_n)^* ,$$

and so the natural linear involution is continuous.

We conclude this section with a generalisation of a result of Moore [59], but first we need to consider the numerical range of elements of A'', the complex unital Banach non-associative algebra under Arens multiplication.

THEOREM 2·1·8. If $F \in A''$, $V(A'',F)$ is the closure of $\{F(f) : f \in D(A,1)\}$. Hence $G \in A''$ is Hermitian if and only if $G(f) \in \mathbb{R}$ for all f in $D(A,1)$.

Proof. By Theorem 2·1·4, $V(A'',F) = V(B(A),L_F)$. By [15], Theorem 9·5,

$$\begin{aligned} V(B(A''),L_F) &= \overline{co}\{(L_F G)(f) : \|G\| = \|f\| = G(f) = 1\} \\ &= \overline{co}\{(FG)(f) : \|G\| = \|f\| = G(f) = 1\} \\ &= \overline{co}\{F(G_f) : \|G\| = \|f\| = G(f) = 1\} . \end{aligned}$$

As $G_f \in A'$, and $\|G_f\| \leq \|G\|.\|f\|$, whenever $\|G\| = \|f\| = G(f) = 1$, we have $G_f \in D(A,1)$ as $G_f(1) = G(f_1) = G(f) = 1$. So

$$V(B(A''),L_F) \subseteq \overline{co}\{F(f) : f \in D(A,1)\} .$$

On the other hand, if $f \in D(A,1)$, and $\tilde{f}$ denotes its canonical image in A''', $\tilde{f} \in D(A'',1)$, and so

$$\{F(f) : f \in D(A,1)\} \subseteq V(A'',F) .$$

Finally, as $\{F(f) : f \in D(A,1)\}$ is convex, $V(A'',F)$ is the closure of $\{F(f) : f \in D(A,1)\}$.

THEOREM 2·1·9. The following are equivalent:

(i) $A = \mathrm{Her}A \oplus i\mathrm{Her}A$,

(ii) $H(A') \cap iH(A') = \{0\}$.

Moreover, if either condition holds, then $A'' = \mathrm{Her}A'' \oplus i\mathrm{Her}A''$.

Proof. (i) ⇒ (ii). Let $R(A') = \{f \in A' : f(\mathrm{Her}A) \subseteq \mathbb{R}\}$. By definition of $H(A')$, $H(A') \subseteq R(A')$. Let $f \in R(A')$. By Corollary 2·1·6 , there exists $g_1, g_2 \in H(A')$ such that $f = g_1 + ig_2$. Suppose there exists $a \in A$ such that $g_2(a) \neq 0$. As $A = \mathrm{Her}A \oplus i\mathrm{Her}A$, without loss of generality, we may assume $a \in \mathrm{Her}A$. Then $(f-g_1)(a) \in \mathbb{R}$, while $i^{-1}(f-g_1)(a) = g_2(a) \in \mathbb{R}\setminus\{0\}$, which is a contradiction. Hence $g_2 = 0$, and so $f \in H(A')$. Thus $H(A') = R(A')$, and, from this, it follows that $H(A') \cap iH(A') = \{0\}$.

(ii) ⇒ (i). By Corollary 2·1·6, $A' = H(A') \oplus iH(A')$. Moreover each $f \in A'$ has a unique expression of the form $f = g_1 + ig_2$ such that

$$\|g_j\| \leq 2\sqrt{2}e\|f\| \qquad (j=1,2) .$$

Hence, if $\{h_n\}$ is a sequence in $H(A')$ converging to $h = k_1 + ik_2$, where k_1 and $k_2 \in H(A')$, for each $n \in \mathbb{N}$, we have

$$h - h_n = (k_1 - h_n) + ik_2$$

and so $h_n \to k_1$. Hence $H(A')$ is closed. Moreover, the natural linear involution on A' is continuous. Now, we define a linear involution on A'' by

$$F^*(f) = (F(f^*))^*$$

for $f \in A'$ and $F \in A''$. By the continuity of the natural linear involution on A' , $F^* \in A''$. Also, for all $f \in H(A')$,

$$\tfrac{1}{2}(F+F^*)(f) \in \mathbb{R} \quad \text{and} \quad \frac{1}{2i}(F-F^*)(f) \in \mathbb{R} .$$

As $D(A,1) \subseteq H(A')$, by Theorem 2·1·8 , we have

$$A'' = \mathrm{Her}A'' \oplus i\mathrm{Her}A'' ,$$

and the involution constructed on A'' is the natural linear involution on A'' with respect to the above decomposition.

By the proof of (i) $\Rightarrow$ (ii), it follows that

$$H(A''') = \{\Phi \in A''' : \Phi(F^*) = \overline{\Phi(F)}\} = \{\Phi \in A''' : \Phi(\mathrm{Her}A'') \subseteq \mathbb{R}\}.$$

Let $\tilde{A}$ denote the canonical image of A in A''. To complete the proof, it suffices to show that $\tilde{A}$ is a self-adjoint subalgebra of A''. If there exists $b \in \tilde{A}$ such that $b^* \notin \tilde{A}$, by the Hahn-Banach theorem, there exists $\Phi \in A'''$ with $\Phi(\tilde{A}) = 0$, but $\Phi(b^*) \neq 0$. By Corollary 2·1·6, $\Phi = \Phi_1 + i\Phi_2$ with $\Phi_j \in H(A''')$ $(j = 1,2)$. Let $f_j = \Phi_j|_{\tilde{A}}$ $(j = 1,2)$, and let $f = \Phi|_{\tilde{A}}$. As $f = 0$, and $f_j \in H(A')$, by hypothesis we have $f_1 = f_2 = 0$. Hence $\Phi(b^*) = (\Phi_1 - i\Phi_2)(b) = 0$, which is a contradiction. Hence $\tilde{A}$ is a self-adjoint subalgebra of A'', as required.

Finally, we have shown during the proof of (ii) $\Rightarrow$ (i) that $A'' = \mathrm{Her}A'' \oplus i\mathrm{Her}A''$

We note that, during the above proof, we showed that $H(A')$ is closed. In fact, under the hypothesis of Theorem 2·1·9, it is possible to show that $(\mathrm{Her}A)' = H(A')$, and $(H(A'))' = \mathrm{Her}(A'')$.

2.2 The Numerical Range in Banach Jordan algebras.

Throughout this section, we let A denote a complex unital Banach Jordan algebra. As $P(1,a)$ is a commutative Banach algebra for all a in A, we may use Lemma 2·1·3 to derive results on $V(a)$ from known results on $V(P(1,a),a)$. Our first three results follow from this and the material of [15] and [14] Section 10.

THEOREM 2·2·1. Let $a \in A$. Then

(i) $\sigma(a) \subseteq V(a)$;

(ii) $\sup\{\mathrm{Re}\lambda : \lambda \in V(a)\} = \inf\{\alpha^{-1}(\|1+\alpha a\|-1) : \alpha > 0\} = \lim\limits_{\alpha\to 0}\{\alpha^{-1}(\|1+\alpha a\|-1) : \alpha > 0\}$;

(iii) $\sup\{\mathrm{Re}\lambda : \lambda \in V(a)\} = \sup\{\alpha^{-1}\log\|\exp\alpha a\| : \alpha > 0\} = \lim\limits_{\alpha\to 0}\{\alpha^{-1}\log\|\exp\alpha a\| : \alpha > 0\}$.

COROLLARY 2·2·2. Given $a \in A$, the following are equivalent:

(i) $h \in \mathrm{Her}A$,

(ii) $\lim\limits_{\alpha\to 0}\{\alpha^{-1}(\|1+i\alpha h\|-1) : \alpha \in \mathbb{R}\setminus\{0\}\} = 0$,

(iii) $\|\exp i\alpha h\| = 1$ for all $\alpha \in \mathbb{R}$.

THEOREM 2·2·3. If $h \in \mathrm{Her}A$, $V(h) = \mathrm{co}\,\sigma(h)$, and

$$r(h) = v(h) = \|h\| .$$

One result, [15] Lemma 5·4, which is not easy to generalise states that if B is a complex unital Banach algebra, and a and b are in $\mathrm{Her}B$, then $i(ab-ba) \in \mathrm{Her}B$. This is used in the proof of the Vidav-Palmer theorem for complex unital Banach algebras to show that if $B = \mathrm{Her}B \oplus i\mathrm{Her}B$, then $\mathrm{Her}B$ is a Jordan subalgebra of B. We give a mild generalisation of this in Theorem 2·2·5, but first we show that $\mathrm{Her}A$ is a Lie triple system.

THEOREM 2·2·4. Let a, b and c be in HerA. Then $(a \circ b) \circ c - a \circ (b \circ c) \in \mathrm{Her}A$, and

$$[[L_c, L_a], L_b] \in \mathrm{Her}B(A)$$

Proof. By Theorem 2·1·4, L_a, L_b and L_c are in HerB(A). Hence by [15], Lemma 5·4, $[[L_c,L_a],L_b] \in \mathrm{Her}B(A)$. The remaining part now follows from Theorem 2·1·4, as

$$[[L_c,L_a],L_b] = L_{(a \circ b) \circ c - a \circ (b \circ c)}.$$

THEOREM 2·2·5. Let B be a complex unital Banach algebra, and suppose that J is a closed Jordan subalgebra of B containing the unit. If $J = \mathrm{Her}J \oplus i\mathrm{Her}J$, then HerJ is a real Jordan algebra.

Proof. It suffices to show that if $h \in \mathrm{Her}J$, then $h^2 \in \mathrm{Her}J$. As J is a Jordan algebra, $h^2 = p + iq$ where $p, q \in \mathrm{Her}J$. As $h(p+iq) = (p+iq)h$, by [15], Lemma 5·4,

$$hp - ph = i(qh-hq) \in \mathrm{Her}A \cap i\mathrm{Her}A = \{0\}.$$

Therefore $hp = ph$, and so $qp - pq = i^{-1}(h^2p-ph^2) = 0$. Hence h^2 is "normal", so that, by [15] Theorem 5·14,

$$V(h^2) = \mathrm{co}\,\sigma(h^2) = \mathrm{co}(\sigma(h))^2 \subseteq \mathbb{R} .$$

Thus $h^2 \in \mathrm{Her}J$, as required.

As we were unable to find out if a similar Theorem held for closed Jordan subalgebras of A, we have to assume HerA is a real Jordan subalgebra of A in our Vidav Palmer theorem.

THEOREM 2·2·6. If HerA is a real Jordan algebra, then it is a unital JB-algebra.

Proof. Given $a, b \in \mathrm{Her}A$, as $\sigma(a) \subseteq \mathbb{R}$, we have

$$\|a\|^2 = (r(a))^2 = r(a^2) = \|a^2\|$$

and

$$\sigma(a^2) = (\sigma(a))^2 \subseteq \mathbb{R}^+ .$$

Hence, for all $g \in D(1)$, by Theorem 2·2·3, $g(a^2) \in \mathbb{R}^+$. As $b^2 \in \mathrm{Her}A$, there exists $f \in D(1)$ such that $\|b^2\| = f(b^2)$. Thus

$$\|a^2 + b^2\| \geq |f(a^2+b^2)| = f(a^2) + f(b^2) \geq \|b^2\| .$$

So HerA is a unital JB-algebra.

This allows us to use the results of section 1·3. The next Theorem is a technical result which we use to derive an extension of the Shirshov-Cohn theorem for certain elements of A.

THEOREM 2·2·7. If A is special, and B is a real closed Jordan subalgebra of A such that B is a unital JB-algebra, $A = B \oplus iB$, and, for all x in A,

$$\|x\| \geq \max(\|h\|, \|k\|)$$

where $x = h + ik$ is the standard decomposition of x with respect to B, then A with the natural involution is homeomorphically $*$-isomorphic to a unital JC*-algebra.

Proof. As A is special, B is a special unital JB-algebra, and so, by Corollary 1·3·11, there is an isometrical isomorphism F of B onto a unital JC-algebra J. If K is the JC*-algebra $J \oplus iJ$, and we define $G : A \to K$ by

$$G(h+ik) = F(h) + iF(k) ,$$

for h and k in B, then G is a $*$-isomorphism of A onto K. Let $x \in A$, with standard decomposition $x = h+ik$. As F is an isometry

$$\|G(x)\| = \|G(h) + iG(k)\| \leq \|h\| + \|k\| \leq 2\|x\|$$

and similarly $\|G^{-1}\| \leq 2$.

THEOREM 2·2·8. If B is a real closed Jordan subalgebra of A such that B is a unital JB-algebra, $A = B \oplus iB$, and, for all x in A ,

$$\|x\| \geq \max(\|h\|,\|k\|)$$

where $x = h + ik$ is the standard decomposition of x with respect to B, then, for each y in A , if $*$ is the natural involution on A , $P(1,y,y^*)$ is a special complex unital Banach Jordan algebra such that $B \cap P(1,y,y^*)$ is a unital JB-algebra and

$$P(1,y,y^*) = (B \cap P(1,y,y^*)) \oplus i(B \cap P(1,y,y^*)).$$

Proof. $B \cap P(1,y,y^*)$ is a unital JB-algebra. Given $x \in A$, we have

$$\|x^*\| \leq \tfrac{1}{2}(\|x+x^*\|+\|x-x^*\|) \leq 2\|x\| ,$$

so $*$ is continuous. Hence $P(1,y,y^*)$ is a self-adjoint subalgebra of A , and so $P(1,y,y^*) = B \cap P(1,y,y^*) \oplus i(B \cap P(1,y,y^*))$. Moreover, $B \cap P(1,y,y^*)$ is the closure of the real Jordan subalgebra of B generated by $y + y^*$ and $i(y-y^*)$. Hence by Corollary 1·3·12, $B \cap P(1,y,y^*)$ is special, and so $P(1,y,y^*)$ is also special.

COROLLARY 2·2·9. (i) If B is a real closed Jordan subalgebra of A such that B is a unital JB-algebra, $A = B \oplus iB$ and, for all x in A ,

$$\|x\| \geq \max(\|h\|,\|k\|) ,$$

where $x = h + ik$ is the standard decomposition of x with respect to B , then if $*$ is the natural involution on A , for each y in A, there is a $*$-isomorphism F of $P(1,y,y^*)$ onto a unital JC^*-algebra, such that $\|F\| \leq 2$, and $\|F^{-1}\| \leq 2$.

(ii) If HerA is a real Jordan subalgebra of A , then for all x in HerA $\oplus$ iHerA ,

$$\|x\| \geq \max(\|h\|,\|k\|) ,$$

where $x = h + ik$ is the standard decomposition of x .

(iii) If HerA is a real Jordan subalgebra of A, and $A = \mathrm{HerA} \oplus i\mathrm{HerA}$, then if $*$ is the natural involution on A, for each y in A, there exists a homeomorphic $*$-isomorphism of $P(1,y,y^*)$ onto a unital JC^*-algebra.

Proof. (i) This follows from Theorems 2·2·7 and 2·2·8.

(ii) For all $f \in D(1)$, if $x = h + ik$ is the standard decomposition of x, we have

$$\|x\| \geq |f(x)| = |f(h) + if(k)| \geq \max(|f(h)|,|f(k)|) .$$

Hence, as h and k are in HerA,

$$\|x\| \geq \max (\|h\| , \|k\|).$$

(iii) This follows from Theorem 2·2·6, and (i) and (ii).

We conclude this section with our first application of Corollary 2·2·9. This result will be generalised in the following sections after we obtain a Russo-Dye theorem and a Vidav-Palmer theorem for A. First we require a Lemma, which, in the finite dimensional case at least, is well known, [17].

LEMMA 2·2·10. Let K be a complex Banach algebra with unit, and let J be a closed Jordan subalgebra of K containing the unit. Then, for all a and b in J, we have

$$\{\exp ia,b,\exp ia\} = (\exp 2iL_a)(b) .$$

Proof. Here, $L_a(b) = \frac{1}{2}(ab+ba)$, and, by writing down the power series expansion for $\exp 2iL_a$, it is clear that, for b in J, $(\exp 2iL_a)(b)$ does not depend on whether we regard L_a as an element of $B(J)$ or $B(K)$. Thus, for the remainder of the proof, we assume $L_a \in B(K)$.

Put $h = ia$, and define m_h and p_h by

$$m_h(x) = hx \quad , \quad p_h(x) = xh$$

for x in K . Then $2iL_a = m_h + p_h$, and m_h commutes with p_h , so, for all b in J , we have

$$\begin{aligned}(\exp 2iL_a)(b) &= \exp(m_h+p_h)b) \\ &= \exp m_h \exp p_h(b) \\ &= (\exp h)b(\exp h) \\ &= \{\exp ia, b, \exp ia\} \quad .\end{aligned}$$

THEOREM 2·2·11. If $A = \mathrm{Her}A \oplus i\mathrm{Her}A$, and $\mathrm{Her}A$ is a real Jordan algebra, then, for all h and k in $\mathrm{Her}A$, and all x in $P(1,h,k)$, we have

(i) $\|\{\exp ih, x, \exp ih\}\| = \|x\|$;

(ii) $\|\{x, \exp ih, x\}\| \leq \|x\|^2$.

Proof. (i) By Corollary 2·2·9, $\bar{P}(1,h,k)$ is homeomorphically *-isomorphic to a unital JC*-algebra. Hence, for all x in $P(1,h,k)$, by Lemma 2·2·10,

$$\{\exp ih, x, \exp ih\} = (\exp 2iL_h)(x) \quad .$$

As $h \in \mathrm{Her}P(1,h,k)$, $L_h \in \mathrm{Her}B(P(1,h,k))$, and so, by Corollary 2·2·2,

$$\|\{\exp ih, x, \exp ih\}\| = \|x\| \quad ,$$

for all x in $P(1,h,k)$.

(ii) As $\{v,\{w,v^2,w\},v\} = \{v,w,v\}^2$ for all v and w in $P(1,h,k)$, for all x in $P(1,h,k)$ we have

$$\begin{aligned}\|\{x, \exp ih, x\}\| &= \|\{\exp\tfrac{1}{2}ih, \{x, \exp ih, x\}, \exp\tfrac{1}{2}ih\}\| \\ &= \|\{\exp\tfrac{1}{2}ih, x, \exp\tfrac{1}{2}ih\}^2\| \\ &\leq \|\{\exp\tfrac{1}{2}ih, x, \exp\tfrac{1}{2}ih\}\|^2 \\ &= \|x\|^2 \quad ,\end{aligned}$$

using (i) twice.

2·3 JB*-algebras.

DEFINITION. Let A be a complex Banach Jordan algebra with an involution * . A is a JB*-algebra if

(i) $\|\{z,z^*,z\}\| = \|z\|^3$ for all z in A ;

(ii) $\|z\| = \|z^*\|$ for all z in A ;

(iii) each norm closed associative *-subalgebra of A is a B*-algebra.

These axioms were introduced by Kaplansky. Actually, both conditions (ii) and (iii) are redundant, the latter being noticed by Wright in [93] .

LEMMA 2·3·1. Let A be a complex Banach Jordan algebra with an involution * , such that for all z in A ,

$$\|\{z,z^*,z\}\| = \|z\|^3 .$$

Then A is a JB*-algebra.

Proof.(a) First, we note that, for all z in A , we have

$$\|z\| \leq 3\|z^*\| \qquad (\dagger)$$

as $\|z\|^3 = \|\{z,z^*,z\}\| \leq 3\|z\|^2\|z^*\|$. Now suppose that, for all z in A,

$$\|z\| \leq \beta\|z^*\| .$$

Then $\|z\|^3 = \|\{z,z^*,z\}\| \leq \beta\|\{z^*,z,z^*\}\| \leq \beta\|z^*\|^3$ for all z in A, and so

$$\|z\| \leq \beta^{1/3}\|z^*\|$$

for all z in A . By induction, starting from $(\dagger)$, we conclude that

$$\|z\| \leq \|z^*\|$$

for all z in A . However, as $z^{**} = z$ for all z in A , it follows that

$$\|z\| = \|z^*\|$$

for all z in A.

(b) Let B be a norm closed associative $*$-subalgebra of A. Then, as $\{z,z^*,z\} = (z \circ z^*) \circ z$, for all z in B, we have

$$\|z\|^3 = \|\{z,z^*,z\}\| \leq \|z \circ z^*\|\|z\| \leq \|z\|^2\|z^*\| = \|z\|^3$$

for all z in B. Thus $\|z \circ z^*\| = \|z\|^2$, and so B is a B^*-algebra.

COROLLARY 2·3·2 A JC^*-algebra is a JB^*-algebra.

Proof. If A is a JC^*-algebra, then A is a complex Banach Jordan algebra with involution, such that, for all x in A,

$$\|\{x,x^*,x\}\|^2 = \|xx^*x\|^2 = \|(xx^*)^3\| = \|xx^*\|^3 = \|x\|^6 .$$

Hence A is a JB^*-algebra.

We shall see later that, as for JB-algebras, the converse to Corollary 2·3·2 is false.

THEOREM 2·3·3. Let A and B be JB^*-algebras, and let f be a $*$-homomorphism of A into B. Then $\|f\| \leq 1$, and f is an isometry if and only if $\operatorname{Ker} f = \{0\}$.

Proof. (i) Let a be a self-adjoint element of A. By Lemma 1·2·1, $P(a)$ is a B^*-algebra, and $R(a)$, the closure of the image of $P(a)$ under f is also a B^*-algebra. Hence, by [69] Corollary 1·2·6,

$$\|f(a)\| \leq \|a\| .$$

For any $x \in A$, let $x = h + ik$ be the standard decomposition of x. Then, as

$$\|h\| = \tfrac{1}{2}\|h+ik+h-ik\| \leq \tfrac{1}{2}(\|x\|+\|x^*\|) = \|x\| ,$$

and similarly $\|k\| \leq \|x\|$, we have

$$\|f(x)\| \leq \|f(h)\| + \|f(k)\| \leq \|h\| + \|k\| \leq 2\|x\| .$$

Hence, f is continuous. Also,

$$\|f(x)\|^3 = \|\{f(x),(f(x))^*,f(x)\}\| = \|f\{x,x^*,x\}\| \leq \|f\|\|x\|^3 ,$$

and so $\|f\| \leq 1$.

(ii) If f is an isometry, then $\mathrm{Ker} f = \{0\}$. Conversely, suppose that $\mathrm{Ker} f = \{0\}$. Then, for each self-adjoint a in A , f is an isomorphism of $P(a)$ into $R(a)$, and hence, by [69], Corollary 1·2·6, $\|f(a)\| = \|a\|$.

For any $x \in A$, let $x = h + ik$ be the standard decomposition of x . As

$$\|x\| \leq \|h\| + \|k\| = \|f(h)\| + \|f(k)\| \leq 2\|f(h+ik)\| = 2\|f(x)\| ,$$

it follows that $\mathrm{Im} f$ is closed. As f is a *-isomorphism, $\mathrm{Im} f$ is thus a JB*-algebra, and $f^{-1}|_{\mathrm{Im}f}$ is a *-isomorphism of $\mathrm{Im} f$ onto A. Hence, by (i), $\|f^{-1}|_{\mathrm{Im}f}\| \leq 1$, and so f is an isometry.

In Chapter 4 , we shall prove a partial converse to Theorem 2·3·3. Now, however, we restrict attention to unital JB*-algebras. It is easily seen that if a JB*-algebra has a unit c , then c must be self-adjoint, and $\|c\| = 1$. The next Theorem is the easy part of the Vidav-Palmer theorem for complex unital Banach Jordan algebras, and it allows us to use the machinery developed in Section 2·2.

THEOREM 2·3·4. (i). If A is a unital JB*-algebra, and a is a self-adjoint element of A , then $a \in \mathrm{Her}A$.

(ii) If A is a complex unital Banach Jordan algebra with continuous involution $*$, such that the self-adjoint elements of A are Hermitian, then $\mathrm{Her}A$ is a JB-algebra, and $A = \mathrm{Her}A \oplus i\mathrm{Her}A$.

(iii) If A is a unital JB*-algebra, then $A = \mathrm{Her}A \oplus i\mathrm{Her}A$, and $\mathrm{Her}A$ is a JB-algebra.

<u>Proof</u>.(i) If a is a self-adjoint element of A , $P(1,a)$ is a

B^*-algebra, and so $\|\exp i\alpha a\| = 1$ for all $\alpha \in \mathbb{R}$. Thus $a \in \mathrm{Her}A$.

(ii). If x is self adjoint, then $x \in \mathrm{Her}A$. Conversely, suppose $x \in \mathrm{Her}A$. If $x = p + iq$ is the standard decomposition of x with respect to the involution $*$, then

$$x - p = iq \in \mathrm{Her}A \cap i\mathrm{Her}A = \{0\} .$$

So $x = p$ is self-adjoint. Thus x is self-adjoint if and only if $x \in \mathrm{Her}A$. Thus $A = \mathrm{Her}A \oplus i\mathrm{Her}A$, and $\mathrm{Her}A$ is a real Jordan algebra. It follows from Theorem 2·2·6 that $\mathrm{Her}A$ is a JB-algebra.

(iii) This follows from (i) and (ii).

THEOREM 2·3·5. (i) If A is a special unital JB^*-algebra, then A is isometrically $*$-isomorphic to a unital JC^*-algebra.

(ii) Let A be a unital JB^*-algebra, and let $x \in A$. Then $P(1,x,x^*)$ is a special unital JB^*-algebra.

Proof (i) By Theorems 2·2·7 and 2·3·4 (iii), and Corollary 2·2·9, A is homeomorphically $*$-isomorphic to a unital JC^*-algebra. By Theorem 2·3·3, this must be an isometry.

(ii) This follows from Theorems 2·3·4 (iii), and 2·2·8.

We conclude this section with a Russo-Dye theorem for unital JB^*-algebras, and one of its applications. For ease of exposition, we introduce the following notation.

Notation Let A be a complex unital Banach Jordan algebra. We let

$$E(A) = \{\exp(ih) : h \in \mathrm{Her}A\} .$$

THEOREM 2·3·6. Let A be a unital JB^*-algebra, and let W be the open unit ball of A. Then

$$W \subseteq \text{co } E(A) \subseteq \overline{\text{co}}\, E(A) = A_1 .$$

Proof. Clearly $0 \in \text{co } E(A)$. Let $x \in W \setminus \{0\}$. By Theorem 2·3·5, $P(1,x,x^*)$ is isometrically isomorphic to a unital JC^*-algebra. If h is a self-adjoint element of $W \cap P(1,x,x^*)$, then $h \in \text{co } E(P(1,x,x^*))$ by [69] Proposition 1·4·5. Hence, if $y \in P(1,x,x^*)$, and $\|y\| < \frac{1}{2}$ and $y = h + ik$ is the standard decomposition of y, then $\|h\| < \frac{1}{2}$, $\|k\| < \frac{1}{2}$, and so $y \in \text{co } E(P(1,x,x^*))$.

Next, as $x \in \|x\|\overline{\text{co}}\, E(P(1,x,x^*))$ by Theorem 1·4·9, there exists $t \in \text{co } E(P(1,x,x^*))$ such that

$$\| \|x\|^{-1}x - t\| < \tfrac{1}{2}(\|x\|^{-1}-1) .$$

Hence, by the above, there exists $z \in \text{co } E(P(1,x,x^*))$ such that $\|x\|^{-1}x - t = (\|x\|^{-1}-1)z$, and so

$$x = \|x\|t + (1 - \|x\|)z \in \text{co } E(P(1,x,x^*)).$$

Thus $W \subseteq \text{co } E(A)$.

It is clear that $\text{co } E(A) \subseteq \overline{\text{co}}\, E(A)$, while, as A_1 is closed and convex and $E(A) \subseteq A_1$, we have $\overline{\text{co}}\, E(A) \subseteq A_1$. Conversely, as $W \subseteq \text{co } E(A)$, we have

$$A_1 = \overline{W} \subseteq \overline{\text{co}}\, E(A) .$$

Thus $A_1 = \overline{\text{co}}\, E(A)$.

THEOREM 2·3·7. Let A be a unital JB^*-algebra.

(i) If $h \in \text{Her}A$, and $y \in A$, then $\|\{\exp ih, y, \exp ih\}\| = \|y\|$.

(ii) For all x, y and z in A, $\|\{x,y,z\}\| \leq \|x\|\,\|y\|\,\|z\|$.

Proof. (i) Let $h \in \text{Her}A$, and $y \in A$. Let $\eta \in \mathbb{R}^+ \setminus \{0\}$. By Theorem 2·3·6, there exists $n \in \mathbb{N}$, $\lambda_j \in \mathbb{R}^+$, and $h_j \in \text{Her}A$ $(j = 1,2,\ldots,n)$ such that $\sum_{j=1}^{n} \lambda_j = \|y\|$ and

$$\|y - \sum_{j=1}^{n} \lambda_j \exp ih_j\| < \eta\|y\| .$$

For each integer j between 1 and n, by Theorems 2·3·4 and 2·2·11,

$$\|\{\exp ih, \exp ih_j, \exp ih\}\| = \|\exp ih_j\| = 1 .$$

Thus

$$\|\{\exp ih, y, \exp ih\}\| \leq \|\{\exp ih, \sum_{j=1}^{n} \lambda_j \exp ih_j, \exp ih\}\| + 3\eta\|y\|$$

$$\leq \sum_{j=1}^{n} \lambda_j \|\{\exp ih, \exp ih_j, \exp ih\}\| + 3\eta\|y\|$$

$$\leq (1 + 3\eta)\|y\| .$$

As this holds for all $\eta \in \mathbb{R}^+ \setminus \{0\}$,

$$\|\{\exp ih, y, \exp ih\}\| \leq \|y\| .$$

Moreover, as $-h \in \mathrm{HerA}$, and $U_{\exp ih} U_{\exp -ih} = I$,

$$\|y\| = \|\{\exp -ih, \{\exp ih, y, \exp ih\}, \exp -ih\}\|$$

$$\leq \|\{\exp ih, y, \exp ih\}\|$$

$$\leq \|y\| .$$

Thus $\|\{\exp ih, y, \exp ih\}\| = \|y\|$.

(ii) As in (i), for any $\eta \in \mathbb{R}^+ \setminus \{0\}$, there exists $n \in \mathbb{N}$, $\lambda_j \in \mathbb{R}^+$ and $h_j \in \mathrm{HerA}$ such that $\sum_{j=1}^{n} \lambda_j = \|y\|$, and $\|y - \sum_{j=1}^{n} \lambda_j \exp ih_j\| < \eta\|y\|$.

The identity $\{v, \{w, v^2, q\}, v\} = \{v, w, v\} \circ \{v, q, v\}$ is valid in any special Jordan algebra, and hence, by Macdonald's Theorem in any Jordan algebra. Thus, for each positive integer j between 1 and n we have

$$\|\{x, \exp ih_j, z\}\| = \|\{\exp \tfrac{1}{2} ih_j, \{x, \exp ih_j, z\}, \exp \tfrac{1}{2} ih_j\}\|$$

$$= \|\{\exp \tfrac{1}{2} ih_j, x, \exp \tfrac{1}{2} ih_j\} \circ \{\exp \tfrac{1}{2} ih_j, z, \exp \tfrac{1}{2} ih_j\}\|$$

$$\leq \|\{\exp \tfrac{1}{2} ih_j, x, \exp \tfrac{1}{2} ih_j\}\| \|\{\exp \tfrac{1}{2} ih_j, z, \exp \tfrac{1}{2} ih_j\}\|$$

$$= \|x\| . \|z\| .$$

Thus

$$\|\{x, y, z\}\| \leq \|\{x, \sum_{j=1}^{n} \lambda_j \exp ih_j, z\}\| + 3\eta\|x\| . \|y\| . \|z\|$$

$$\leq \sum_{j=1}^{n} \lambda_j \|\{x, \exp ih_j, z\}\| + 3\eta\|x\| . \|y\| . \|z\|$$

$$\leq (1 + 3\eta)\|x\| . \|y\| . \|z\| .$$

As this holds for all $\eta \in \mathbb{R}^+ \setminus \{0\}$, $\|\{x, y, z\}\| \leq \|x\| \|y\| \|z\|$.

2.4 The Vidav-Palmer theorem for Banach Jordan algebras.

We only require one last Lemma before the Vidav Palmer theorem for complex unital Banach Jordan algebras.

LEMMA 2·4·1. Let A be a complex unital Banach Jordan algebra such that $A = \mathrm{Her}A \oplus i\mathrm{Her}A$, and $\mathrm{Her}A$ is a real Jordan algebra. Let B be a unital JB^*-algebra, and let $G : A \to B$ be a homeomorphic *-isomorphism, where A is given the natural involution. Then, for all a in A,

$$\|a\| \leq \|G(a)\| .$$

Proof. It suffices to show that if $a \in A$ and $\|G(a)\| = 1$, then $a \in A_1$. If $\|G(a)\| = 1$, then by Theorem 2·3·6, there is a sequence $\{b_n\}$ in $\mathrm{co}\, E(B)$ such that

$$\|b_n - G(a)\| \to 0 .$$

As G is a homeomorphic *-isomorphism, if $c_n = G^{-1}(b_n)$ for all n, then $c_n \in \mathrm{co}\, E(A)$ and

$$\|c_n - a\| \to 0 .$$

In particular, $\|c_n\| \to \|a\|$. However, as $E(A) \subseteq A_1$, for all $n \in \mathbb{N}$, $\|c_n\| \leq 1$. Thus $a \in A_1$, as required.

THEOREM 2·4·2. Let A be a complex unital Banach Jordan algebra. A is a JB^*-algebra if and only if $A = \mathrm{Her}A \oplus i\mathrm{Her}A$ and $\mathrm{Her}A$ is a real Jordan algebra.

Proof. If A is a JB^*-algebra, then, by Theorem 2·3·4(iii), $A = \mathrm{Her}A \oplus i\mathrm{Her}A$, and $\mathrm{Her}A$ is a real Jordan algebra.

Conversely, suppose that $A = \mathrm{Her}A \oplus i\mathrm{Her}A$ and $\mathrm{Her}A$ is a real Jordan algebra. Fix $z \in A$. By Corollary 2·2·9, there is a homeomorphic *-isomorphism G of $P(1,z,z^*)$ onto a unital JC*-algebra. Thus by Theorem 2·2·11, for all x in $P(1,z,z^*)$ and all h in $\mathrm{Her}P(1,z,z^*)$, we have

$$\|\{x,\exp ih,x\}\| \leq \|x\|^2 \quad .$$

Now let $\omega, \eta \in \mathbb{R}^+ \setminus \{0\}$. Pick $a \in P(1,z,z^*)$ such that $\|a\| = 1$, and $\|G(a)\| \geq \|G\| - \eta$. As $\|G(a^*)\| = \|(G(a))^*\| = \|G(a)\|$, we have

$$G(a^*) \in \|G(a)\| \ \overline{\mathrm{co}}\ E(G(P(1,z,z^*)))$$

by Theorem 2·3·6. As G is a homeomorphic *-isomorphism, this implies

$$a^* \in \|G(a)\| \ \overline{\mathrm{co}}\ E(P(1,z,z^*)) \ .$$

Pick $y \in \|G(a)\| \ \mathrm{co}\ E(P(1,z,z^*))$ such that $\|a^* - y\| < \omega$. Then, there exists $n \in \mathbb{N}$, $h_j \in \mathrm{Her}P(1,z,z^*)$ and $\lambda_j \in \mathbb{R}^+$ $(j = 1,2,\ldots,n)$, such that $\sum_{j=1}^{n} \lambda_j = \|G(a)\|$, and $y = \sum_{j=1}^{n} \lambda_j \exp ih_j$. Therefore

$$\begin{aligned}
\|\{a,a^*,a\}\| &\leq \|\{a,y,a\}\| + 3\|a\|^2\omega \\
&\leq \Big(\sum_{j=1}^{n} \lambda_j \|\{a,\exp ih_j,a\}\|\Big) + 3\omega \\
&\leq \sum_{j=1}^{n} \lambda_j \|a\|^2 + 3\omega \\
&= \|G(a)\| + 3\omega \\
&\leq \|G\| + 3\omega \quad .
\end{aligned}$$

So

$$\begin{aligned}
\|G(a)\|^3 &= \|G(\{a,a^*,a\})\| \\
&\leq \|G\| \ \|\{a,a^*,a\}\| \\
&\leq \|G\| \ (\|G\| + 3\omega)
\end{aligned}$$

As this holds for all $\omega > 0$,

$$\|G(a)\|^3 \leq \|G\|^2 \quad .$$

However, $\|G(a)\| > \|G\| - \eta$, so that

$$(\|G\| - \eta)^3 \leq \|G\|^2 \ .$$

As this holds for all $\eta > 0$, $\|G\|^3 \leq \|G\|^2$, and so $\|G\| \leq 1$.

However, by Lemma 2·4·1, for all x in $P(1,z,z^*)$, we have

$$\|x\| \leq \|G(x)\| \ .$$

Combining these results, we conclude that G is an isometry, and so,

$$\|\{z,z^*,z\}\| = \|G\{z,z^*,z\}\| = \|G(z)\|^3 = \|z\|^3 \ .$$

Thus A is a JB^*-algebra.

The remainder of this section is devoted to applications of the Vidav Palmer Theorem. We start with analogues of results in [15], Section 7.

THEOREM 2·4·3. Let A be a complex unital Banach Jordan algebra, with a continuous involution $*$, such that, for all x in A ,

$$\|\{x,x^*,x\}\| = \|x\|^2\|x^*\| \ .$$

Then A is a JB^*-algebra.

Proof. Let $h \in A$ be self-adjoint. As, for all y in $P(1,h)$, $\{y,y^*,y\} = (yoy^*)oy$, it follows that

$$\|y\|^2\|y^*\| = \|(yoy^*)oy\| \leq \|yoy^*\| \ \|y\| \leq \|y\|^2\|y^*\| \ .$$

Thus $\|yoy^*\| = \|y\| \ \|y^*\|$, and so, by [34] or [15], Theorem 7·2, $P(1,h)$ is a unital B^*-algebra. Hence $h \in \mathrm{Her}P(1,h)$. So, by Theorems 2·4·2 and 2·3·4, A is a JB^*-algebra.

THEOREM 2·4·4. Let A be a complex unital Banach Jordan algebra, with a continuous involution $*$ such that every associative self-adjoint subalgebra of A is a B^*-algebra. Then A is a JB^*-algebra.

Proof. Let $h \in A$ be self-adjoint. As $*$ is continuous, $P(1,h)$ is an associative subalgebra of A. Hence $P(1,h)$ is a B^*-algebra, and and so $h \in \mathrm{Her}A$. It again follows by Theorems 2·4·2 and 2·3·4 that A is a JB^*-algebra.

THEOREM 2·4·5. Let A be a complex unital Banach Jordan algebra with a continuous involution $*$, such that, for all self-adjoint h in A, and for all λ in $\mathbb{C}$, $\sigma(h) \subseteq \mathbb{R}$, and $r(1+\lambda h) = \|1+\lambda h\|$. Then A is a JB*-algebra.

Proof. Let $k \in A$ be self-adjoint. Then $P(1,h)$ is a commutative unital complex Banach $*$-algebra such that, for all self-adjoint k in $P(1,h)$, and for all λ in $\mathbb{C}$, $\sigma(k) \subseteq \mathbb{R}$, and $r(1+\lambda k) = \|1+\lambda k\|$. By [15], Theorem 7·4, $P(1,h)$ is a B^*-algebra, so $h \in \mathrm{Her}A$. By Theorems 2·4·2, and 2·3·4, A is a JB*-algebra.

THEOREM 2·4·6. Let A be a complex unital Banach Jordan algebra, and suppose that HerA is a real Jordan algebra. Then $\mathrm{Her}A \oplus i\mathrm{Her}A$ is a JB*-algebra, and if B is a closed subalgebra of A containing the unit that is a JB*-algebra with respect to some involution, then $B \subseteq \mathrm{Her}A \oplus i\mathrm{Her}A$.

Proof. This follows from Lemma 2·1·3, and Theorem 2·4·2.

The next Theorem has no natural analogue in the theory of Banach algebras. It is a converse to Theorem 2·3·4, and was also obtained by Wright [93].

THEOREM 2·4·7. Let B be a unital JB-algebra, and let $A=B\oplus iB$ be the complexification of B. Then there is an equivalent norm on A such that A, with the natural involution, is a unital JB*-algebra.

Proof. We recall from Lemma 1·2·8 that there is a norm p on A such that (A,p) is a complex unital Banach Jordan algebra, the natural involution on A is an isometry,

$$2\max(\|a\|,\|b\|) \geq p(a+ib) \geq \max(\|a\|,\|b\|)$$

and $p(a) = \|a\|$ for all a and b in B.

Let $V = \mathrm{co}\{\exp ih; h \in B\}$. If $x = \sum_{j=1}^{n} a_j \exp ih_j$ with $a_j \in \mathbb{R}^+$, $h_j \in B$ and $r = \sum_{j=1}^{n} a_j \leq 1$, then

$$x = \sum_{j=1}^{n} a_j \exp ih_j + \tfrac{1}{2}(1-r)\exp i0 + \tfrac{1}{2}(1-r)\exp i\pi \in V .$$

Hence $V = \{\sum_{j=1}^{n} a_j \exp ih_j : n \in \mathbb{N},\ a_j \in \mathbb{R}^+,\ \sum_{j=1}^{n} a_j \leq 1, \text{ and } h_j \in B\}$, so V is balanced and convex.

Now suppose $p(x) < \frac{1}{2}$. By Corollary 2·2·9, there is a homeomorphic *-isomorphism F from $P(1,x,x^*)$ onto a JC^*-algebra such that $\|F(x)\| < 1$. Hence, by Theorem 2·3·6, $F(x) \in \mathrm{co}\, E(P(F(1),F(x),(F(x))^*))$. As F is a homeomorphic *-isomorphism, $x \in V$, and so V is absorbing.

Let μ be the Minkowski functional of V . Then μ is a seminorm on A such that $p(x) < \frac{1}{2}$ implies $\mu(x) < 1$. Conversely, if $\mu(x) < 1$, then $x \in V$, and so, for some $n \in \mathbb{N}$,

$$x = \sum_{j=1}^{n} a_j \exp ih_j \quad \text{with} \quad a_j \in \mathbb{R}^+,\ \sum_{j=1}^{n} a_j \leq 1, \text{ and } h_j \in B.$$

Hence

$$\begin{aligned} p(x) &= p(\sum_{j=1}^{n} a_j \cos h_j + i \sum_{j=1}^{n} a_j \sin h_j) \\ &\leq 2\max(\|\sum_{j=1}^{n} a_j \cos h_j\|, \|\sum_{j=1}^{n} a_j \sin h_j\|) \\ &\leq 2 . \end{aligned}$$

Thus, μ is an equivalent linear space norm on A, and in particular, (A,μ) is a Banach space. Moreover, if $h \in B$, and $h = t\sum_{s=1}^{m} b_s \exp ik_s$ where $t \in \mathbb{R}^+$, $b_s \in \mathbb{R}^+$, $\sum_{s=1}^{m} b_s \leq 1$, and $k_s \in B$, then

$$h = \tfrac{1}{2}(h+h^*) = t\sum_{s=1}^{m} b_s \cos k_s$$

so

$$\|h\| \leq t\|\sum_{s=1}^{m} b_s \cos k_s\| \leq t .$$

Thus $\mu(h) \geq \|h\|$. Conversely, if $h \in B$ and $\|h\| < 1$, then, by Theorems 1·3·2 and 2·3·6, $h \in V$, and so $\mu(h) < 1$. Thus, if $h \in B$,

$$\mu(h) = \|h\| .$$

To show that (A,μ) is a Banach Jordan algebra, it suffices to show that if $x, y \in V$, then $x \circ y \in qV$ for all $q \in \mathbb{R}$ such that $q > 1$. Hence, suppose $m, n \in \mathbb{N}$, $a_j, c_k \in \mathbb{R}^+$ $(1 \leq j \leq n)$, $(1 \leq k \leq m)$, such that $\sum_{j=1}^{n} a_j = \sum_{k=1}^{m} c_k = 1$, and $h_j, t_k \in B$ $(1 \leq j \leq n), (1 \leq k \leq m)$. If $\alpha = \sum_{j=1}^{n} a_j \exp ih_j$ and $\beta = \sum_{k=1}^{m} c_k \exp it_k$, then $\alpha \circ \beta = \sum_{j=1}^{n} \sum_{k=1}^{m} a_j c_k \exp ih_j \circ \exp it_k$, with $a_j c_k \in \mathbb{R}^+$, and $\sum_{j=1}^{n} \sum_{k=1}^{m} a_j c_k = 1$. Moreover, for all $q \in \mathbb{R}$ with $q > 1$, by Corollary 2·2·9, and Theorem 2·3·6,

$$\exp ih_j \circ \exp it_k \in qV .$$

Thus $\alpha \circ \beta \in qV$ for all $q > 1$, as required.

It is clear that $\mu(a) = \mu(a^*)$ for all a in A, and that $\mu(\exp ih) \leq 1$ for all h in B. Hence, for all h in B, we have

$$1 = \|1\| = \mu(1) = \mu((\exp ih) \circ \exp(-ih)) \leq (\mu(\exp ih))^2 \leq 1 ,$$

so that, for all h in B, $\mu(\exp ih) = 1$. By Theorem 2·4·2, (A,μ) is a JB^*-algebra.

One immediate conclusion of Theorem 2·4·7 is that there exists non-special JB^*-algebras. Moreover, by Theorems 2·4·7 and 1·3·10, the following Gelfand Neumark theorem holds for unital JB^*-algebras. (This was also obtained in [93]) .

THEOREM 2·4·8. Let A be a unital JB^*-algebra. Then there is a family F of Jordan homomorphisms $A \to \operatorname{Im} f$ of norm at most one, such that

(i) for all $f \in F$, either $\operatorname{Im} f$ is a JC^*-algebra, or $\operatorname{Im} f$ is the complexification of M_3^8 with norm given by Theorem 2·4·7;

(ii) for all non-zero $a \in A$, there exists $f \in F$ such that $f(a) \neq 0$.

As for unital JB-algebras, Theorem 2·4·8 may be restated as follows. The proof is essentially due to Shultz [76].

THEOREM 2·4·9. Let A be a unital JB^*-algebra. Then A is isometrically *-isomorphic to a self-adjoint Jordan subalgebra of $C(X, M_3^8 \oplus iM_3^8) \oplus B(\mathcal{H})$ where X is a compact Hausdorff space, and $\mathcal{H}$ is a complex Hilbert space.

Proof. Let F be the family of Jordan homomorphisms given in Theorem 2·4·8. Let $I = \{f \in F : \mathrm{Im} f \text{ is a } JC^*\text{-algebra}\}$, and let $J = F \setminus I$. If, for each $f \in I$, $\mathrm{Im} f \subseteq B(\mathcal{H}_f)$, then $\sum_{f \in I} \mathrm{Im} f \subseteq B(\mathcal{H})$, where $\mathcal{H} = \bigoplus_{f \in I} \mathcal{H}_f$. If X is the Stone-Cech compactification of the discrete set J, then $\sum_{f \in J} \mathrm{Im} f = C(X, M_3^8 \oplus iM_3^8)$. Hence, by Theorem 2·4·8, A is isometrically *-isomorphic to a self-adjoint Jordan subalgebra of $C(X, M_3^8 \oplus iM_3^8) \oplus B(\mathcal{H})$.

COROLLARY 2·4·10. If A is a unital JB^*-algebra, then the following are equivalent:

(i) A is isometrically *-isomorphic to a JC^*-algebra,

(ii) A is special,

(iii) If $a, b, c \in \mathrm{Her} A$, and f is any s-identity for M_3^8, then $f(a,b,c) = 0$,

(iv) $\mathrm{Her} A$ is special.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i) If A is not isometrically *-isomorphic to a JC^*-algebra, there is a non-zero Jordan *-homomorphism from A onto $M_3^8 \oplus iM_3^8$. Hence, there is a non-zero Jordan homomorphism g of $\mathrm{Her} A$ onto M_3^8. If f is an s-identity, and x, y, z are elements of M_3^8 such that $f(x,y,z) \neq 0$, then, given any elements a, b and c in

HerA such that $x = g(a)$, $y = g(b)$ and $z = g(c)$, we have

$$0 = f(a,b,c) = g(f(a,b,c)) = f(x,y,z) \neq 0 \quad ,$$

which is a contradiction.

As we noted after Theorem 2·2·4 , we do not know if HerA is a real Jordan algebra even if HerA$\oplus$iHerA is a complex Jordan algebra. However, Theorem 2·2·5 gives a hypothesis under which this happens, and this gives the following result.

COROLLARY 2·4·11. Let B be a complex unital Banach algebra, and suppose that $A = \mathrm{Her}B \oplus i\mathrm{Her}B$ is a closed Jordan subalgebra of B . Then A is isometrically *-isomorphic to a JC*-algebra.

Proof. This follows from Theorems 2·2·5 and 2·4·6 and Corollary 2·4·10.

We now turn out attention to quotient algebras of unital JB*-algebras. A well known theorem states that every closed, two-sided ideal of a B*-algebra is self-adjoint, see, for example, [69] Corollary 1·17·3. This was extended to closed Jordan ideals by Civin and Yood [24], and we now give a further extension to closed ideals of a unital JB*-algebra by modifying tha proof of Theorem 7·7 in [15].

THEOREM 2·4·12. Let A be a unital JB*-algebra, and let I be a proper closed ideal of A . Then

(i) if $h \in \mathrm{Her}A$, then $h + I \in \mathrm{Her}A/I$, and so $A/I = \mathrm{Her}A/I \oplus i\mathrm{Her}A/I$;

(ii) if $a \in I$, then $a^* \in I$;

(iii) Her(A/I) is a real Jordan algebra;

(iv) A/I is a unital JB*-algebra.

Proof. (i) By Theorem 1·2·9, A/I is a complex unital Banach Jordan algebra. Given $f \in D(A/I,1+I)$, we define $g \in A'$ by

$$g(a) = f(a+I) ,$$

for a in A . Then $g(1) = 1$, while

$$|g(a)| = |f(a+I)| \leq \|f\| \|a+I\| \leq \|a\| .$$

So $g \in D(A,1)$, and hence if $h \in \mathrm{Her}A$, then $f(h+I) = g(h) \in \mathbb{R}$. Thus $h + I \in \mathrm{Her}A/I$. Hence $A/I = \mathrm{Her}A/I \oplus i\mathrm{Her}A/I$.

(ii) Let $a \in I$, and let $a = h + ik$ be the standard decomposition of a. If $h \notin I$, by (i), $h + I$ is a non-zero element of $\mathrm{Her}A/I$, and so there exists $f \in D(A/I,1+I)$ such that $f(h+I) \in \mathbb{R}\setminus\{0\}$. Again by (i), $f(k+I) \in \mathbb{R}$, and so

$$0 = f(a+I) = f(h+I) + if(k+I) \neq 0 ,$$

which is a contradiction. Thus $h \in I$, and so $k \in I$. Hence $a^* = h - ik \in I$.

(iii) . Suppose that $b + I \in \mathrm{Her}A/I$. If $p + iq$ is the natural decomposition of b , then, by (i) ,

$$(b-p) + I = iq + I \in \mathrm{Her}A/I \cap i\,\mathrm{Her}A/I = \{0\} ,$$

and so $q \in I$. Hence,

$$(b+I)^2 = (p+I)^2 = p^2 + I \in \mathrm{Her}A/I .$$

Thus $\mathrm{Her}A/I$ is a real Jordan algebra.

(iv). This follows by (i), (iii) and Theorem 2·4·2.

COROLLARY 2·4·13. Let A be a unital JC^*-algebra, and let I be a closed ideal of A . Then A/I is isometrically *-isomorphic to a JC^*-algebra.

Proof. Let f be an s-identity for M_3^8 . If $b + I \in \mathrm{Her}A/I$, we showed in the proof a Theorem 2·4·12 that there exists $h \in \mathrm{Her}A$ such that $b + I = h + I$. It follows that, whenever $a + I$, $b + I$ and $c + I \in \mathrm{Her}A/I$, then $f(a+I,b+I,c+I) = 0$. Hence, by Theorem 2·4·12 and Corollary 2·4·10, A/I is isometrically *-isomorphic to a unital

JC^*-algebra.

The above Corollary was essentially obtained in [31] by more elementary means. Corollary 2·4·13 and part of Theorem 2·4·12 were also proved in [93]. The following Corollary also appears in [93], and is an easy extension of [5] Theorem 9.5. As we shall not require it later, we do not give the proof.

COROLLARY 2·4·14. Let A be a unital JB^*-algebra. Then there is a closed Jordan ideal I in A such that A/I is special, and such that whenever K is any closed Jordan ideal of A with A/K special, then $I \subseteq K$.

We conclude this section with a generalisation of Theorem 2·3·3, and an application of Theorem 2·1·9.

THEOREM 2·4·15. Let A and B be unital JB^*-algebras, and let f be a *-homomorphism of A into B. Then $\mathrm{Im}f$ is closed, and f' is an isometry of $(\mathrm{Im}f)'$ into A'.

Proof. By Theorem 2·3·3, f is continuous, so $\mathrm{Ker}f$ is a closed ideal of A. By Theorem 2·4·12, $J = A/\mathrm{Ker}f$ is a JB^*-algebra. Let $k : A \to J$ be the canonical map, and let $g : J \to B$ be defined by $g(k(a)) = f(a)$. By Theorem 1·1·1, g is a *-isomorphism of J onto $\mathrm{Im}f$, and so, by Theorem 2·2·3, g is an isometry. Hence $\mathrm{Im}f$ is closed, and, moreover, g' is an isometry of $(\mathrm{Im}f)'$ onto J'. As k' is also an isometry by normed linear space theory, we conclude that $f' = k'g'$ is an isometry of $(\mathrm{Im}f)'$ into A'.

THEOREM 2·4·16. Let A be a complex unital Banach Jordan algebra such that $\mathrm{Her}A$ is a real Jordan algebra. Then the following are equivalent:

(i) A is a JB^*-algebra,

(ii) $A = \mathrm{Her}A \oplus i\mathrm{Her}A$,

(iii) $H(A') \cap iH(A') = \{0\}$.

Proof. (i) $\Leftrightarrow$ (ii) by Theorem 2·4·2, while (ii) $\Leftrightarrow$ (iii) by Theorem 2·1·9.

CHAPTER 3

We study two main topics in this chapter. The first is the problem of renorming $A \oplus \mathbb{C}$, where A is a non-unital JB*-algebra, and $A \oplus \mathbb{C}$ is the Banach Jordan algebra with product and norm given in Lemma 1·2·7, so that $A \oplus \mathbb{C}$ is a JB*-algebra in the new norm. We could not do this in general, but we were able to do it in a special case. Our solution uses a generalisation of results of Bonsall, [11], [13]. These give conditions on a real unital Banach Jordan algebra, A , under which A is a JB-algebra in an equivalent norm. An alternative approach is due to Smith [78], but he was also unable to solve the problem completely.

The second main topic is the study of JB*- algebras which are Banach dual spaces. Part of their importance is due to their links with quantum mechanics, see [4], and the earliest study in this area was prompted by these links [44], [61]. Work in this area has been rather sporadic since then, however, although Segal [73] introduced an axiom scheme for what are now called Segal algebras, and which extracted many of the essential ideas in [61]. Recently Deliyannis, [26], showed that a Segal algebra satisfying an additional axiom may be embedded into a larger Segal algebra which has "many" projections, the construction being similar to [5] Chapter 3. An argument used in [3] p.113 then shows that the larger Segal algebra is a JB-algebra. A different approach, which comes to the same conclusion is given in [3] and [4].

Some of the theory for JB-algebras which are Banach dual spaces is given in [76]. JB*-algebras which are Banach dual spaces are the Jordan analogues of W*-algebras, and some of the theory we give is a generalisation of material in [69]. However, the last section is devoted to a generalisation of results of Topping [84], in particular the concepts of centre, annihilators and quadratic ideals.

3·1. JB-algebras and JB*-algebras in an equivalent norm.

This section is concerned with conditions under which a real or complex unital Banach Jordan algebra is respectively a JB-algebra or a JB*-algebra in an equivalent norm. The main result for JB-algebras is a generalisation of work of Bonsall [11], [13], and we derive the result on JB*-algebras using Theorem 2·4·7.

Throughout this section, we let A be a complex unital Banach Jordan algebra.

Notation We let $\mathrm{splsa}A = \{a \in A : \sigma(a) \subseteq \mathbb{R}^+\}$ and $\mathrm{Inv}A = \{a \in A : 0 \notin \sigma(a)\}$.

DEFINITION. Let Q be a subset of A such that $1 \in Q$.

(i) Q is a wedge if, for all x and y in Q and $\alpha \in \mathbb{R}^+$, αx and $x + y$ are in Q.

(ii) Q is a local semialgebra if, for all x and y in Q and $\alpha \in \mathbb{R}^+$ such that $(P(1,x,y),\circ)$ is a commutative Banach algebra, then αx, $x + y$ and $x \circ y$ are in Q.

(iii) Q is locally multiplicative if $x, y \in Q$ implies $\{x,y,x\} \in Q$.

(iv) For each $n \in \mathbb{P}$, Q is type n if $x \in Q$ implies $x^n \circ (1+x)^{-1} \in Q$.

LEMMA 3·1·1. If Q is a subset of A such that $1 \in Q$, and if either Q is locally multiplicative, or Q is a local semialgebra, then $x \in Q$ implies $x^n \in Q$ for all $n \in \mathbb{P}$.

Proof. This follows by induction.

THEOREM 3·1·2. For any $n \in \mathbb{P}$, splsaA is a type n local semialgebra which contains all type 0 local semialgebras and type 0 locally multiplicative wedges.

Proof. Clearly $1 \in$ splsaA, and if $a \in$ splsaA and $r \in \mathbb{R}^+$, then $ra \in$ splsaA. By Theorem 1·1·8, splsaA is type n for all $n \in \mathbb{P}$. Suppose $x,y \in$ splsaA such that $(P(1,x,y),o)$ is a commutative Banach algebra. By Corollary 1·2·6, $x,y \in$ splsaP(1,x,y), and so $x + y$ and $x \circ y$ are in splsaP(1,x,y) $\subseteq$ splsaA.

Now suppose that Q is either a type 0 local semialgebra or a type 0 locally multiplicative wedge. As Q is type 0 and positive homogeneous, $\alpha + x$ is invertible, and $(\alpha+x)^{-1}$ is invertible for all $\alpha \in \mathbb{R}^+$ and all $x \in Q$. So

$$x \in Q \text{ implies } \sigma(x) \cap (-\mathbb{R}^+) \subseteq \{0\} . \qquad (\dagger)$$

Suppose $a \in Q$ and $r \exp i\theta \in \sigma(a)$ with $r \in \mathbb{R}^+ \setminus \{0\}$ and $0 < |\theta| < \pi$. Let $n \in \mathbb{N}$ be such that $\pi/2 \leqslant n|\theta| < \pi$, and let $b = a^n$. By Lemma 3·1·1, $b \in Q$, and $\sigma(b)$ contains a point $-\gamma + i\delta$ with $\gamma \in \mathbb{R}^+$ and $\delta \in \mathbb{R}\setminus\{0\}$. Then $(\gamma+b)^2 \in Q$, and $-\delta^2 \in \sigma((\gamma+b)^2)$, which contradicts $(\dagger)$. Hence $\sigma(a) \in \mathbb{R}^+$, as required.

LEMMA 3·1·3. Let Q be a closed type 0 local semialgebra, or a closed type 0 locally multiplicative wedge in A. If $a \in Q$ and $\alpha \geqslant r(a)$, then $\alpha - a \in Q$.

Proof. Let $c \in Q$ with $r(c) < 1$. Then $(1-c)^{-1} = 1 + b$ where $b = \sum_{k=1}^{\infty} c^k \in Q$. Hence $1 - c = (1+b)^{-1} \in Q$. So, by a suitable scalar multiplication, if $a \in Q$ and $\alpha > r(a)$, then $\alpha - a \in Q$. As Q is closed, it follows that $r(a) - a \in Q$.

For the remainder of the chapter, we let W be a closed, type 0 locally multiplicative wedge in A, and let R be the real linear span of W in A. The next three results give some of the properties of R and W.

THEOREM 3·1·4. (i) $W = R \cap \mathrm{splsa}A$.

(ii) $a \in R$ implies $\sigma(a) \subseteq \mathbb{R}$.

(iii) R is a real closed Jordan subalgebra of A containing the unit.

(iv) If a and b are in R, then $r(a+b) \leq r(a) + r(b)$.

(v) Conversely if S is a subset of A satisfying (ii),(iii) and (iv), and $X = S \cap \mathrm{splsa}A$, then X is a closed type 0 locally multiplicative wedge, and S is the real linear span of X.

Proof. (i) and (ii) Let $a = x - y$ with $x, y \in W$, and let $\alpha = r(y)$. As $a + \alpha = x + (\alpha - y) \in W$, we have $\sigma(a+\alpha) \subseteq \mathbb{R}^+$, and so $\sigma(a) \subseteq \mathbb{R}$.

Moreover, if $\sigma(a) \subseteq \mathbb{R}^+$, then $r(a+\alpha) = r(a) + \alpha$, and so

$$r(a) - a = r(a+\alpha) - (a+\alpha) \in W.$$

As $\sigma(a) \subseteq [0, r(a)]$, it follows that $r(r(a) - a) \leq r(a)$, and so

$$a = r(a) - (r(a)-a) \in W.$$

Hence, as W is a wedge, $R \cap \mathrm{splsa}A \subseteq W$. Conversely, by Theorem 3·1·2, $W \subseteq R \cap \mathrm{splsa}A$. Hence $W = R \cap \mathrm{splsa}A$.

(iii) As W is a wedge, R is a real linear subspace of A containing the unit. Let $\{a_n\}$ be a sequence in R which converges to $a \in A$. Choose $m \in \mathbb{N}$ such that, if $p, q \geq m$, then $\|a_p - a_q\| \leq 1$. As $\sigma(a_p - a_m) \subseteq \mathbb{R}$, $1 + (a_p - a_m) \in W$ for $p \geq m$.

As W is closed, $1 + a - a_m \varepsilon W$, and so $a \varepsilon R$. Thus R is closed.

Now suppose $b \varepsilon R$ and $b = x - y$, where $x, y \varepsilon W$. Let $z = x + y$. By Lemma 3·1·1, x^2, y^2 and z^2 are in W, so that

$$a^2 = x^2 + y^2 - 2x \circ y = x^2 + y^2 - z^2 + x^2 + y^2 \varepsilon R .$$

Hence R is a Jordan subalgebra of A.

(iv) If $a, b \varepsilon R$, then $r(a) \pm a$ and $r(b) \pm b$ are in W. Thus $r(a) + r(b) \pm (a+b) \varepsilon W$ and so

$$r(a+b) \leq r(a) + r(b) .$$

(v) Let $u, v \varepsilon X$, and $\lambda \varepsilon \mathbb{R}^+$. Clearly $\lambda v \varepsilon X$. Also

$$r(r(u) + r(v) - (u+v)) \leq r(r(v)-v) + r(r(u) - u) \leq r(u) + r(v)$$

and since $\sigma(u+v) \subseteq \mathbb{R}$, it follows that $\sigma(u+v) \subseteq \mathbb{R}^+$. Hence $u + v \varepsilon X$.

Now suppose $\{a_n\}$ is a sequence in X which converges to $a \varepsilon A$. As S is closed, $a \varepsilon S$. Also, for all $n \varepsilon \mathbb{N}$,

$$\begin{aligned} r(\|a\|-a) &\leq |\,\|a\|-\|a_n\|\,| + r(\|a_n\|-a_n) + r(a_n-a) \\ &\leq |\,\|a\|-\|a_n\|\,| + \|a_n\| + \|a_n - a\| \end{aligned}$$

and hence

$$r(\|a\|-a) \leq \|a\| .$$

Thus $\sigma(a) \subseteq \mathbb{R}^+$, and so $a \varepsilon X$. Hence X is closed.

Let $u \varepsilon X$. Then $(1+u)^{-1} \varepsilon$ splsaA, and if $\alpha > 1 + r(u)$ and $x = \alpha^{-1}(\alpha-1-u)$, then $r(x) < 1$ and $1 + u = \alpha(1-x)$. So $(1+u)^{-1} = \alpha^{-1} \sum_{n=0}^{\infty} x^n \varepsilon S$. Thus X is type 0.

Finally, let $u, v \varepsilon X$, and let $\eta \varepsilon \mathbb{R}^+\setminus\{0\}$. As $\eta + u \varepsilon \operatorname{Inv} A$, and $(\eta+u)^{-2} \varepsilon X \cap \operatorname{Inv} A$, there exists $w \varepsilon X$ and $\delta \varepsilon \mathbb{R}^+\setminus\{0\}$ such that $(\eta+u)^{-2} = \delta + w$. $w + v \varepsilon X$, and so $\delta + v + w \varepsilon \operatorname{Inv}A \cap X$. Hence $x = (\eta+u)^{-2} + v \varepsilon X \cap \operatorname{Inv}A$, and so, as S is a Jordan algebra,

$$\{\eta + u, x, \eta + u\} \varepsilon S \cap \operatorname{Inv}A ,$$

that is, $1 + \{\eta+u, v, \eta+u\} \varepsilon S \cap \operatorname{Inv}A$. As this is true for all v in X, and $\sigma(\{\eta+u,v,\eta+u\}) \subseteq \mathbb{R}$, we conclude that $\sigma(\{\eta+u,v,\eta+u\}) \subseteq \mathbb{R}^+$. Hence $\{\eta+u,v,\eta+u\} \in X$, and as X is closed, it follows that $\{u,v,u\} \in X$.

THEOREM 3·1·5 (i) $a \in R$ implies $a^2 \in W$.

(ii) $r(aob) \leq r(a)r(b)$ for all a and b in R.

(iii) $W \cap(-W) = \{a \in R : r(a) = 0\}$ is an ideal of R .

(iv) If $w \in W \cap \mathrm{Inv}A$, there exists $x \in P(1,w) \cap W \cap \mathrm{Inv}A$ such that $x^2 = w$.

Proof . (i) This is immediate from Theorem 3·1·4.

(ii) Let $a, b \in R$, and let $t \in \mathbb{R}$. By (i), $(a-tb)^2 \in W$. As $r(a^2) - a^2$ and $r(b^2) - b^2$ are both in W , and W is a wedge,

$$r(a^2) - 2t.a \circ b + t^2 r(b^2) \in W .$$

Hence, if $\lambda \in \sigma(aob)$, we have

$$r(a^2) - 2\lambda t + t^2 r(b^2) \in \mathbb{R}^+ .$$

As this holds for all $t \in \mathbb{R}$, it follows that $\lambda^2 \leq r(a^2)r(b^2)$. Hence $|\lambda| \leq r(a)r(b)$, and so

$$r(aob) \leq r(a)r(b) .$$

(iii) If $a \in W \cap(-W)$, then $\sigma(a) \subseteq \mathbb{R}^+ \cap(-\mathbb{R}^+) = \{0\}$, and so $r(a) = 0$. Conversely, if $a \in R$, and $r(a) = 0$, then $\sigma(\pm a) \subseteq \mathbb{R}^+$, and so $a \in W \cap(-W)$. $W \cap(-W)$ is a closed real linear subspace of R , and by (ii) , $W \cap(-W)$ is an ideal.

(iv) Let $w \in W \cap \mathrm{Inv}A$. Without loss of generality, we may assume $r(w) < 1$. Then $r(1-w) < 1$, and so, as R is a Jordan algebra, there exists $x \in R \cap P(1,w) \cap \mathrm{Inv}A$ such that $x^2 = w$, and $\sigma(x) \subseteq \mathbb{R}^+$. Hence $x \in W \cap \mathrm{Inv}A \cap P(1,w)$.

LEMMA 3·1·6. If $x \in R$ and $w \in W$, then $\{x,w,x\} \in W$.

Proof. Let $\eta \in \mathbb{R}^+ \setminus \{0\}$, and let $c = w + \eta$. Then $c \in W \cap \mathrm{Inv}A$, so there exists $b \in W \cap \mathrm{Inv}A$ such that $b^2 = c$. By Macdonald's theorem,

$$U_b U_x(b^2) = \{b,x,b\}^2 ,$$

so, by Theorem 3·1·4, we have $U_b U_x c \in W$. As $b \in \mathrm{Inv}A$, there exists $d \in A$ such that $U_d U_b = I$, and moreover, if $b = \delta + a$ where $a \in W$ and $\delta \in \mathbb{R}^+ \setminus \{0\}$, then

$$d = \delta^{-1}(1+\delta^{-1}a)^{-1} \in W .$$

As W is locally multiplicative, $U_x c = U_d U_b U_x c \in W$. Finally, as W is closed, it follows that $\{x,w,x\} \in W$.

In order to derive deeper results, we require a strong boundedness condition. Alternative characterisations of this are given in the following Theorem, and from this we derive a characterisation of those real Jordan subalgebras of A which are unital JB-algebras in an equivalent norm.

THEOREM 3·1·7. The following conditions are equivalent :

(i) $W \cap (-W) = \{0\}$ and there exists $M \in \mathbb{R}^+$ such that for all w in W, $\|(1+w)^{-1}\| \leq M$,

(ii) W is a normal cone, that is, there exists $\kappa \in \mathbb{R}^+ \setminus \{0\}$ such that if x and y are in W, then

$$\|x+y\| \geq \kappa\|x\| ,$$

(iii) there exists $\kappa \in \mathbb{R}^+ \setminus \{0\}$ such that, for all x in R,

$$r(x) \geq \kappa\|x\| .$$

Proof. (i) $\Rightarrow$ (iii). Suppose $x \in R$ and $r(x) = \frac{1}{2}$. Then $\sigma(1+x) \subseteq [1/2, 3/2]$, and so $(1+x)^{-1} \in W$ and $\sigma((1+x)^{-1}) \subseteq [2/3, 2]$. Hence $(1+x)^{-1} = 2/3 + w$ for some $w \in W$, and $1 + x = 3/2(1+3/2 w)^{-1}$.

Thus

$$\|x\| \leqslant (1+\|1+x\|) \leqslant {}^3/_2 M+1 \leqslant (3M+2)r(x) .$$

Hence, by a suitable scalar multiplication, as $r(y) > 0$ for all $y \in W\setminus\{0\}$ by Theorem 3·1·5,

$$\|y\| \leqslant (3M+2)r(y)$$

for all y in R.

(iii) $\Rightarrow$ (ii). Let $x, y \in W$. As $r(x+y)-(x+y) \in W$, it follows that $r(x+y) - x \in W$, and so $r(x) \leqslant r(x+y)$. Thus

$$\|x+y\| \geqslant r(x+y) \geqslant r(x) \geqslant \kappa\|x\|$$

(ii) $\Rightarrow$ (i). Given $w \in W$, there exists $x \in W \cap P(1,(1+w)^{-1})$ such that $x^2 = (1+w)^{-1}$. As $w \in W$, $(1+w)^{-1} \in P(1,w)$, and hence $P(1,(1+w)^{-1}) \subseteq P(1,w)$. Thus

$$\{x,1+w,x\} = (1+w) \circ x^2 = 1 ,$$

and hence, as x^2 and $\{x,w,x\}$ are in W, we have

$$1 = \|1\| = \|\{x,1+w,x\}\| = \|x^2+\{x,w,x\}\| \geqslant \kappa\|x^2\| = \kappa\|(1+w)^{-1}\| .$$

Hence $\|(1+w)^{-1}\| \leqslant \kappa^{-1}$. Moreover, if $x \in W \cap(-W)$, then $0 = \|x+(-x)\| \geqslant \kappa\|x\|$, so $x = 0$.

COROLLARY 3·1·8. If any of the equivalent conditions in Theorem 3·1·7 hold, then r is a norm on R such that (R,r) is a JB-algebra and such that r and $\|.\|$ are equivalent on R.

<u>Proof</u>. By Theorems 3·1·4, 3·1·5 and 3·1·7 (R,r) is a real Banach Jordan algebra such that r and $\|.\|$ are equivalent on R. Clearly $r(x^2) = (r(x))^2$, and in the proof of Theorem 3·1·7 (iii) $\Rightarrow$ (ii), we showed that

$$r(x) \leqslant r(x+y)$$

for x and y in W. As the squares of elements of R are in W, we conclude that (R,r) is a JB-algebra.

COROLLARY 3·1·9. If any of the equivalent conditions in Theorem 3·1·7 hold, then $W \cap (-W) = \{0\}$, and each $w \in W$ has a unique square root in W.

Proof. This is an immediate consequence of Corollary 3·1·8, Lemma 1·3·1 and Corollary 1·3·4..

Remark The hypothesis that $\{(1+w)^{-1} : w \in W\}$ is bounded implies that $W \cap -W = \{0\}$, as shown in [11] Theorem 9. We shall not require this.

COROLLARY 3·1·10. Let S be a real Jordan subalgebra of A containing the unit. S is a JB-algebra in an equivalent norm if and only if

(i) S is closed,

(ii) $\sigma(x) \subseteq \mathbb{R}$ for all x in S,

(iii) $r(x+y) \leq r(x) + r(y)$ for all x and y in S,

(iv) there exists a constant $\kappa \in \mathbb{R}^+ \setminus \{0\}$ such that, for all x in S,

$$r(x) \geq \kappa \|x\| .$$

Proof. If S satisfies conditions (i), (ii), (iii) and (iv), then, by Theorem 3·1·4 and Corollary 3·1·8, S is a JB-algebra in an equivalent norm.

Conversely suppose that p is an equivalent norm on S under which S is a JB-algebra. As, for all x in S,

$$r(x) = \lim_{n\to\infty} \|x^{2^n}\|^{2^{-n}} = \lim_{n\to\infty} p(x^{2^n})^{2^{-n}} = p(x) ,$$

(i), (iii) and (iv) are clear. By Lemma 1·3·1, Corollary 1·3·4 and Theorem 1·3·5, S^2 is a locally multiplicative wedge, and so (ii) follows by Theorem 3·1·4.

We now turn to a characterisation of JB^*-algebras in an equivalent norm. We first require a Lemma.

LEMMA 3·1·11. Let $*$ be a continuous involution on A such that S, the set of self-adjoint elements of A, is a JB-algebra in an equivalent norm, p. If B is the complexification of S with a norm q such that (B,q) is a JB*-algebra, and $q(s) = p(s)$ for all s in S, then there is a homeomorphic *-isomorphism from A onto B.

Proof If $F : B \to A$ is defined by $F((a,b)) = a + ib$ for a and b in S, then F is a Jordan *-isomorphism of B onto A. By hypotheses, there exists $\kappa \in \mathbb{R}^+$ such that $\|x\| \leq \kappa p(x)$ for all x in S. Thus

$$\|F(a,b)\| = \|a+ib\| \leq \kappa(p(a)+p(b) \leq 2\kappa q(a,b) .$$

Hence, F is continuous, and by the open mapping theorem, F is a homeomorphism.

THEOREM 3·1·12. If $*$ is a continuous involution on A such that, for some $\kappa \in \mathbb{R}^+$, with $\kappa \geq 1$,

(i) $\kappa^{-1} \leq \|\exp ih\| \leq \kappa$ for all self-adjoint h in A;

(ii) $r(h+k) \leq r(h) + r(k)$ for all self-adjoint h and k in A;

then A is a JB*-algebra in an equivalent norm.

Proof. Let S be the closed real Jordan subalgebra of self-adjoint elements of A. Suppose that there exists $h \in S$ and $\alpha, \beta \in \mathbb{R}$ such that $\beta \neq 0$, and $\alpha + i\beta \in \sigma(h)$. Let $k = \beta^{-1}(h-\alpha)$, so that $k \in S$ and $i \in \sigma(k)$. Hence, for all $\eta \in \mathbb{R}^+$, $\exp \eta \in \sigma_{P(1,k)}(\exp(-i\eta k))$. So, for all $\eta \in \mathbb{R}^+$,

$$\exp \eta \leq \|\exp(-i\eta k)\| \leq \kappa .$$

This is a contradiction, and so, if $h \in S$, then $\sigma(h) \subseteq \mathbb{R}$.

We now show, for all h in S,

$$r(h) \geq \kappa^{-2} \|h\| .$$

This is a well known result on renorming Hermitian equivalent elements, see [14] Theorem 5·1. Let $h \in S$. Then $P(1,h)$ is a complex unital commutative Banach algebra with continuous involution which inherits properties (i) and (ii) from A. We define two new norms on $P(1,h)$ as follows:

$$\mu(x) = \sup\{\|(\exp ith)o\, x\| : t \in \mathbb{R}\}$$

and
$$p(x) = \sup\{\mu(xoy) : \mu(y) \leq 1\} .$$

It is easy to check that μ and p are norms on $P(1,h)$ such that

$$\kappa^{-1}\|x\| \leq \mu(x) \leq \kappa\|x\|$$

and
$$\kappa^{-2}\|x\| \leq p(x) \leq \kappa^{2}\|x\|$$

for all x in $P(1,h)$. Moreover $(P(1,h)\, ,\, p)$ is a commutative unital Banach algebra, so it follows that the spectral radius is the same in either norm. Given $t \in \mathbb{R}$,

$$\mu((\exp ith)o\, y) = \sup\{\|(\exp i(t+s)h)o\, y\| : s \in \mathbb{R}\} = \mu(y) ,$$

and so $p(\exp ith) = 1$. Thus $h \in Her(P(1,h),p)$, and so

$$p(h) = r(h) \geq \kappa^{-2}\|h\| .$$

By Corollary 3·1·10, S is a JB-algebra in an equivalent norm. Hence, by Lemma 3·1·11, and Theorem 2·4·7, A is a JB*-algebra in an equivalent norm.

We note that the converse to Theorem 3·1·12 is clear. When A is a Jordan subalgebra of a complex unital Banach *-algebra, an easier proof of Theorem 3·1·12 may be extracted from [72].

3.2 Adjoining a unit to a JB*-algebra.

Up to this point, the JB-algebras and JB*-algebras we have considered have generally been unital. In this section we consider the problem of adjoining a unit to a non-unital JB*-algebra and renorming to get a unital JB*-algebra. We cannot follow the approach adopted in [69] for B*-algebras, as it relies on the left regular representation being a homomorphism. In Chapter 5, we shall indicate how this may be modified for Jordan algebras, but we take an alternative approach here.

Throughout this section, we let A be a JB*-algebra which does not have a unit, and we let B be the set of self-adjoint elements of A. We remark that it is not clear if B is a JB-algebra. We let $J = A \oplus \mathbb{C}$ be the complex unital Banach Jordan algebra with norm and product given in Lemma 1·2·7, and extend the involution on A to a continuous involution on J defined by

$$(a,\mu)^* = (a^*,\bar{\mu})$$

for a in A and μ in $\mathbb{C}$. We let $K = B \oplus \mathbb{R}$ be the set of self-adjoint elements of J.

LEMMA 3·2·1. J is a complex unital Banach Jordan algebra with continuous involution, such that, for all $h \in K$,

$$1 \leq \|\exp ih\| \leq 3 .$$

Proof. We have only to prove the last inequality. Let $h \in K$, and suppose $h = b + \lambda$ where $b \in B$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} \|\exp ih\| &= \|\exp i\lambda \,.\, \exp ib\| \\ &= \|\exp ib\| \\ &= 1 + \|\sum_{n=1}^{\infty} (n!)^{-1}(ib)^n\| . \end{aligned}$$

By Lemma 2·3·1, $P(b)$ is a non-unital commutative B*-algebra, and so

$$0 \leq \|\sum_{n=1}^{\infty}(n!)^{-1}(ib)^n\| \leq 1+1 .$$

Hence

$$1 \leq \|\exp ih\| \leq 3 .$$

By Theorem 3·1·12, to show that J is a JB*-algebra in an equivalent norm, it suffices to show that the spectral radius is subadditive on elements of K. Unfortunately, we were unable to do this in general, so we have to put an additional hypothesis on elements of B. First, we clear up a possible ambiguity about notation.

A positive element of a JB*-algebra Y is a self-adjoint element of Y which is the square of a self-adjoint element of Y. We denote by Y^+ the set of positive elements of Y. The set of positive functionals on Y, which we denote by $(Y')^+$ is $\{f \in Y' : f(y) \in \mathbb{R}^+ \text{ for all } y \text{ in } Y^+\}$. If Z is the set of self-adjoint elements of Y, and $f \in (Y')^+$, then $f|_Z \in Z'$. Conversely, as

$$\|x\| \leq \|x+iy\| \leq \|x\| + \|y\|$$

for all x and y in Z, it follows that for every $g \in Z'$ such that $g(x) \in \mathbb{R}^+$ for all $x \in Y^+$, there exists $h \in (Y')^+$ such that $h|_Z = g$.

Also, we note that by the uniqueness of the positive square root in a B*-algebra, and Lemma 2·3·1, $Y^+ \cap (-Y^+) = \{0\}$.

THEOREM 3·2·2. Suppose that, for all d in B, $\inf\{\|a^2+d\| : a^2 + d \in A^+\} = \|d^+\|$, where d^+ is the unique positive square root of d. Then, for all x and y in B,

(i) $\|x^2-y^2\| \leq \max(\|x\|^2,\|y\|^2)$;

(ii) A^+ is a closed convex cone ;

(iii) $\|x^2+y^2\| \geq \|x\|^2$.

Proof. Given $x, y \in B$, we let $c = x^2 - y^2$. As $(-c)^+ = c^-$, we have

$$\|c^+\| \leq \|(x^2-y^2) + y^2\| = \|x^2\| ,$$

$$\|c^-\| \leq \|(y^2-x^2) + x^2\| = \|y^2\| ,$$

and so

$$\|c\| = \max(\|c^+\|,\|c^-\|) \leq \max(\|x^2\|,\|y^2\|).$$

Let $Q = \{a \in B : \|a\| \leq 1 \text{ and } \|b^2-a\| \leq 1 \text{ for all } b \text{ in } B_1\}$. Q is a closed, convex set. If $a \in A^+ \setminus \{0\}$, then $\|a\|^{-1}a \in Q$, as

$$\|b^2 - \|a\|^{-1}a\| \leq \max(\|b^2\|,1) = 1$$

for b in B_1, while conversely, if $a \in Q$ and $\|a\| = 1$, then $a \in A^+$. Hence, as $0 \in Q \cap A^+$, it follows that A^+ is a convex cone. To show A^+ is closed, we argue as follows. Let $\{a_n\}$ be a sequence in A^+ such that $a_n \to a$. If $a = 0$, then $a \in A^+$, so we may assume $a \neq 0$. Then $\|a_n\|^{-1}a_n \to \|a\|^{-1}a$, and so, as Q is closed, $\|a\|^{-1}a \in Q$. Hence $\|a\|^{-1}a \in A^+$, and so $a \in A^+$.

As $(b^2)^+ = b^2$ for all b in B, it follows that

$$\|x^2+y^2\| \geq \|x\|^2 .$$

Starting with the axiom $\|a^2-b^2\| \leq \max(\|a\|^2,\|b\|^2)$ for a and b in B, Smith [73] is able to show that K is a JB-algebra in an equivalent norm. Our proof of Theorem 3·2·2 was greatly simplified by the use of the set Q which was introduced in [78]. The crucial step in the argument in [78] is the existance of sufficient positive linear functionals of norm one. These play the role of the states in a unital JB*-algebra, as we now see.

THEOREM 3·2·3. Let Y be a JB*-algebra, and let Z be the set of self-adjoint elements of Y.

(i) If Y is unital, then $f \in D(Y,1)$ if and only if $f \in (Y')^+$ and

$\|f|_Z\| = 1$.

(ii). If Y is unital, then, for all d in Z ,

$$\inf\{\|a^2+d\| : a^2+d \in Y^+\} = \|d^+\|.$$

(iii). If Y is such that, $\inf\{\|a^2+d\|:a^2+d \in Y^+\} = \|d^+\|$ for all d in Z, then, whenever N is a closed self-adjoint subalgebra of Y , $f \in (N')^+$ and $\|f|_{N\cap Z}\| = 1$, then there exists $g \in (Y')^+$ such that $\|g|_Z\| = 1$, and $g|_N = f$.

Proof.(i) If $f \in D(Y,1)$, then $f \in (Y')^+$ by Theorem 2·3·4, and clearly $\|f|_Z\| = 1$.

Conversely, suppose that $\|f|_Z\| = 1$, and $f \in (Y')^+$. Let $\eta \in \mathbb{R}^+\setminus\{0\}$, and let $h \in Z_1$ be such that

$$\|f|_Z\| < |f(h)| + \eta .$$

As $1 \pm h \in Y^+$, we have $f(1\pm h) \in \mathbb{R}^+$, and so

$$|f(h)| \leq f(1) .$$

Hence, it follows that $f(1) = 1 = \|f|_Z\|$.

Now let $k \in Z$. $f|_{P(1,k)}$ is a positive linear functional on a unital B^*-algebra, and so, by [14] Lemma 37·6,

$$|f(\exp ik)|^2 \leq f(1)f((\exp ik)o(\exp -ik)) = 1 .$$

Hence, by Theorem 2·3·6, $|f(a)| \leq 1$ for all $a \in Y_1$. Thus $\|f\| = 1$, and so $f \in D(Y,1)$.

(ii) Suppose there exists x and y in Z such that $\|x^2+y\| < \|y^+\|$ and $x^2 + y \in Y^+$. By Lemmas 2·1·3 and 2·3·1, there exists $k \in D(Y,1)$ such that $k(y^+) = \|y^+\|$, and $k(y^-) = 0$. Hence

$$0 \leq k(x^2+y) < k(y^+) .$$

As $x^2 \in Y^+$, it follows that

$$0 \leq k(x^2) < k(y^-) = 0 .$$

This is a contradiction, and so $\|x^2+y\| \geq \|y^+\|$. As $y^+ = y^- + y \in Y^+$, it follows that the infimum is attained.

(iii) An argument similar to Theorem 3·2·2 shows that Y^+ is a closed convex cone. Define $p : Z \to \mathbb{R}^+$ by

$$p(a) = \|a^+\| = \inf\{\|b^2+a\| : b \in Z \text{ and } b^2+a \in Y^+\} .$$

As Y^+ is a cone, $p(ta) = tp(a)$ for all $t \in \mathbb{R}^+$ and $a \in Z$. Moreover, if $a, b, c, d \in Z$ are such that $a^2 + c \in Y^+$ and $b^2 + d \in Y^+$, then $a^2 + b^2 + c + d \in Y^+$ and $a^2 + b^2 \in Y^+$, so that

$$p(c+d) \leq \|a^2+b^2+c+d\| \leq \|a^2+c\| + \|b^2+d\| .$$

Hence

$$p(c+d) \leq p(c) + p(d) .$$

If $f \in (N')^+$ and $\|f|_{N\cap Z}\| = 1$, for any $z \in N \cap Z$ we have

$$f(z) = f(z^+) - f(z^-) \leq f(z^+) \leq \|z^+\| = p(z) .$$

Thus, by the Hahn-Banach theorem, there exists a real linear functional $g : Z \to \mathbb{R}$ such that $-p(-b) \leq g(b) \leq p(b)$ for all b in Z, and $f|_{N\cap Z} = g|_Z$. For $y \in Z$,

$$|g(y)| \leq \max(\|y^+\|, \|y^-\|) = \|y\| ,$$

so $\|g\| = 1$. If in addition, $y \in Y^+$, $p(-y) = 0$, and so $g(y) \in \mathbb{R}^+$. The result follows upon identifying g with an element of $(Y')^+$, as described before Theorem 3·2·2.

THEOREM 3·2·4. Suppose that, for all d in B, and for all $f \in ((P(d))')^+$ such that $\|f|_{P(d)\cap B}\| = 1$, there exists $g \in (A')^+$ such that $g|_{P(d)} = f$ and $\|g|_B\| = 1$.

(i) Given $x \in B$ and $\mu \in \mathbb{R}$, $\sigma(x+\mu) \subseteq \mathbb{R}^+$ if and only if $\mu \in \mathbb{R}^+$ and, for all g in $(A')^+$ with $\|g|_B\| = 1$, $g(x) \geq -\mu$.

(ii). If $h, k \in K$ are such that $\sigma(h) \subseteq \mathbb{R}^+$ and $\sigma(k) \subseteq \mathbb{R}^+$, then $\sigma(h+k) \subseteq \mathbb{R}^+$.

(iii) $r(h+k) \leq r(h) + r(k)$ for all h and k in K.

Proof (i) Let $x \in B$ and $\mu \in \mathbb{R}$. By [69] Proposition 1·1·7,

$P(1,x+\mu)$ may be renormed in an equivalent norm under which it is a commutative B^*-algebra, and we may identify $P(1,x+\mu)$ with $C(\sigma(x+\mu))$. Suppose $x + \mu \in (C(\sigma(x+\mu)))^+$. As $0 \in \sigma(x)$, it follows that $\mu \in \mathbb{R}^+$. Let $g \in (A')^+$ be such that $\|g|_B\| = 1$. Then either $g(x) = 0$, in which case $g(x) \geq -\mu$, or $g|_{P(x)\cap B} \neq 0$. In the latter case, we let $f = \|g|_{P(x)\cap B}\|^{-1} g|_{P(x)}$ and note by Theorem 3·2·3 that there exists $m \in D(C(\sigma(x+\mu), 1)$ such that $m|_{P(x)} = f$. As $x + \mu$ is positive,

$$f(x) = m(x) \geq -\mu ,$$

and so, as $\|g|_{B\cap P(x)}\| \leq \|g|_B\| = 1$, we have

$$g(x) \geq -\|g|_{P(x)\cap B}\|\mu \geq -\mu .$$

Conversely suppose $x + \mu \notin (C(\sigma(x+\mu)))^+$. Then, there is $\lambda \in \mathbb{R}^+ \setminus \{0\}$ such that $-\lambda \in \sigma(x+\mu)$. If $\mu \in \mathbb{R}^+$, there is a multiplicative linear functional $f \in (P(1,x+\mu))'$ such that $f(x) = -\lambda-\mu$, and $\|f|_{B\cap P(x)}\| = 1$. Hence, there exists $g \in (A')^+$ such that $g|_{P(x)} = f|_{P(x)}$ and $\|g|_B\| = 1$. So

$$g(x) = f(x) = -\lambda - \mu < -\mu .$$

(ii) Let $g \in (A')^+$ be such that $\|g|_B\| = 1$. Let $y, z \in B$, $\lambda,\mu \in \mathbb{R}$ be such that $\sigma(h) \subseteq \mathbb{R}^+$ and $\sigma(k) \subseteq \mathbb{R}^+$ where $h = y + \lambda$ and $k = z + \mu$. By (i), λ and μ are in $\mathbb{R}^+$ and $g(y) \geq -\lambda$, and $g(z) \geq -\mu$. Hence $\lambda + \mu \in \mathbb{R}^+$, and $g(y+z) \geq -(\lambda+\mu)$. Again by (i), $\sigma(h+k) \in \mathbb{R}^+$.

(iii) Let $a, b \in K$. As $\sigma(r(a)-a) \subseteq \mathbb{R}^+$ and $\sigma(r(b)-b) \subseteq \mathbb{R}^+$, we have

$$\sigma(r(a) + r(b) - (a+b)) \subseteq \mathbb{R}^+ .$$

Hence $r(a+b) \leq r(a) + r(b)$.

The preceeding results in this section may be conveniently summarised in the following Corollary.

COROLLARY 3·2·5. If either of the following two conditions are satisfied, then J is a JB^*-algebra in an equivalent norm.

(i) For all d in B, $\inf\{\|a^2+d\| : a^2+d \in A^+\} = \|d^+\|$.

(ii) For all d in B and all f in $((P(d))')^+$ such that $\|f|_{P(d)\cap B}\| = 1$, there exists $g \in (A')^+$ such that $g|_{P(d)} = f$ and $\|g|_B\| = 1$.

We conjecture that J is always a JB^*-algebra in an equivalent norm. We conclude this section with a structure Theorem for embedable JB^*-algebras. The proof follows from Theorems 2·4·8 and 2·3·5, the only part perhaps not being immediately clear is that a self-adjoint subalgebra of $M_3^8 \oplus iM_3^8$ is semi-simple by [43] Corollary 5·5·1, and hence has a unit by [43] Corollary 5·5·2.

DEFINITION. A JB^*-algebra Y is called embedable if there is an isometrical *-isomorphism of Y into a unital JB^*-algebra.

THEOREM 3·2·6. Let Y be an embedable JB^*-algebra and let a and b be self-adjoint elements of Y.

(i) $P(a,b)$ is isometrically *-isomorphic to a JC^*-algebra.

(ii) There is a family F of Jordan homomorphisms $f \to \mathrm{Im} f$ of norm at most one, such that, for each non-zero $a \in Y$, there exists $f \in F$ such that $f(a) \neq 0$, and for all $f \in F$, either $\mathrm{Im} f$ is a JC^*-algebra or $\mathrm{Im} f = M_3^8 \oplus iM_3^8$.

3·3. JB*-algebras which are Banach dual spaces.

In the proof of Theorem 3·2·6, we implicitly used the result that any finite dimensional JB*-algebra has a unit. This fails in general, even for B^*-algebras, but it is well known that a von Neumann algebra has a unit. It makes no sense to consider a "weak operator closed" JB*-algebra, but fortunately, non-spatial characterisations of von Neumann algebras exist ([48] and [68]). Shultz showed that two similar characterisations coincide for unital JB-algebras, in [76].

THEOREM 3·3·1. (Shultz). Let A be a unital JB-algebra. A is the dual of a Banach space if and only if A is monotone complete and has a separating set of normal functionals.

Later in the section, we shall prove the easy part of this Theorem, but we are more concerned in this section with the properties of a similar class of JB*-algebra. We start off with JC*-algebras, and an analogue of results of Topping [84], and Effros and Størmer [31].

THEOREM 3·3·2. Let $\mathcal{H}$ be a complex Hilbert space, A a closed self-adjoint Jordan subalgebra of $B(\mathcal{H})$, and let B be the set of self-adjoint elements of A. If J and K are respectively the weak operator closures of A and B, then K is a JC-algebra, and $J = K \oplus iK$ is a JC*-algebra.

Proof. Clearly $K \oplus iK \subseteq J$. As the involution on $B(\mathcal{H})$ is weak operator continuous, conversely $J \subseteq K \oplus iK$, and hence $J = K \oplus iK$. To complete the proof, we shall show that K is a real Jordan subalgebra of self-adjoint operators of $B(\mathcal{H})$, and this will imply that $J = K \oplus iK$ is a JC*-algebra. The argument is due to Topping [84].

K is a closed real linear subspace of self-adjoint operators. As B is a convex set, the strong and weak operator closures of B coincide. Hence, if $a \in K$, there exists a net $\{a_\alpha\}$ in B such that, for all x in $\mathcal{H}$,

$$\|a_\alpha x\| \to \|ax\| .$$

Hence $\|a_\alpha x\|^2 \to \|ax\|^2$ for all x in $\mathcal{H}$, and so

$$(a_\alpha^2 x,x) \to (a^2 x,x)$$

for all x in $\mathcal{H}$. Hence $a_\alpha^2 \to a^2$ in the weak operator topology. As $a_\alpha^2 \in B$ for all α, it follows that $a^2 \in K$, so K is a Jordan algebra.

THEOREM 3·3·3. Let A be a JC^*-algebra. There is an isometrical *-isomorphism F of A'' with Arens multiplication onto a weak operator closed JC^*-algebra such that $a_\alpha \to a$ $(\tau(A'',A'))$ if and only if $F(a_\alpha) \to F(a)$ in the weak operator topology.

Proof. Let B be the C^*-algebra generated by A. By Theorem 1·2·16, there is an isometrical *-isomorphism G of B'' with Arens multiplication onto the weak operator closure K of B in its universal respresentation such that $b_\alpha \to b$ $\tau(B'',B')$ if and only if $G(b_\alpha) \to G(b)$ in the weak operator topology.

Let $j : A \to B$ be the injection map. Then $j'' : A'' \to B''$ is an isometrical *-isomorphism of A'' with Arens multiplication into B'' with Arens multiplication, such that $a_\alpha \to a$ $(\tau(A'',A'))$ if and only if $j''(a_\alpha) \to j''(a)$ $(\tau(B'',B'))$, and $j''(A'')$ is $\tau(B'',B')$ closed. As B is Arens regular, $F = Gj''$ is the map with the required properties.

Our next three results are essentially due to Shultz [76] and Smith [78].

THEOREM 3·3·4. If A is a unital JB^*-algebra, then A'' with Arens multiplication is a unital JB^*-algebra.

Proof By Theorem 2·4·9, A is isometrically *-isomorphic to a self-adjoint Jordan subalgebra of $J = C(X, M_3^8 \oplus iM_3^8) \oplus B(\mathcal{H})$, where X is a compact Hausdorff space, and $\mathcal{H}$ is a complex Hilbert space. By Theorems 1·2·17 and 3·3·3, and Lemma 1·2·13, J'' is a unital JB^*-algebra, where the Jordan product on J'' is the Arens product. If e is the unit of A then e is an idempotent of J'', $U_e(J'')$ is a JB^*-algebra with unit e, and $U_e(J'')$ contains A''. A'' is a Jordan algebra by Lemma 1·2·14, and $A'' = \mathrm{Her}A'' \oplus i\mathrm{Her}A''$ by Theorem 2·1·9. As $\mathrm{Her}A'' = (\mathrm{Her}U_e(J'')) \cap A''$, it follows that $\mathrm{Her}A''$ is a real Jordan algebra. Hence, by Theorem 2·4·2, A'' is a unital JB^*-algebra.

LEMMA 3·3·5. If A is a unital JB^*-algebra, and B is a $\tau(A'',A')$ closed self-adjoint subalgebra of A'', then there exists an idempotent e in B such that e is an identity for B.

Proof. By Theorem 1·3·6, B has an increasing approximate identity $\{e_\lambda\}$ of positive elements of norm at most one. If $r \in \mathbb{R}^+$, and $f \in D(A,1)$, then $\{e_\lambda(rf)\}$ is a bounded increasing net in $\mathbb{R}^+$, and so by Corollary 2·1·6, $\{e_\lambda\}$ is a $\tau(A'',A')$ Cauchy net. Hence $\{e_\lambda\}$ converges to an element $e \in B$ $(\tau(A'',A'))$, and, by Theorem 2·1·8, $e \in \mathrm{Her}A''$, and $e = \sup\{e_\lambda\}$.

By Theorem 1·3·5, for all λ we have

$$\begin{aligned}(e-e_\lambda)^2 &\leq \|e-e_\lambda\|(e-e_\lambda)\\ &\leq 2(e-e_\lambda),\end{aligned}$$

and so $(e-e_\lambda)^2 \to 0$ $(\tau(A'',A'))$. Let $a \in B \cap \mathrm{Her}(A'')$, and let $f \in D(A,1)$. As

$$\begin{aligned} |(a-aoe)(f)| &\leq |(a-aoe_\lambda)(f)| + |ao(e-e_\lambda)(f)| \\ &\leq \|a-aoe_\lambda\| + \left(|a^2(f)|.|(e-e_\lambda)^2(f)|\right)^{\frac{1}{2}} \\ &\to 0 \end{aligned}$$

as $\lambda \to \infty$, $a = a \circ e$ by Corollary 2·1·6. Finally as B is a self-adjoint subalgebra, it follows that e is a unit for B.

THEOREM 3·3·6. If A is an embedable JB^*-algebra, then A'' is a unital JB^*-algebra.

Proof. Let J be a unital JB^*-algebra, and let $j : A \to J$ be an isometrical $*$-isomorphism. If $(j(A))^\perp$ denotes the Banach space annihilator of $j(A)$ in J', see [66], and J' is given the natural linear involution as in Theorem 2·1·9, then it successively follows that $(j(A))^\perp$ and $(j(A))^{\perp\perp}$ are self-adjoint subspaces of J' and J'' respectively. Hence $j'' : A'' \to (j(A))^{\perp\perp}$ is an isometrical $*$-isomorphism of A'' with the Arens product onto the $\tau(J'',J')$ closed self-adjoint Jordan subalgebra $(j(A))^{\perp\perp}$ of J''. Hence A'' is a JB^*-algebra, and A'' has a unit by Lemma 3.3.5.

In the remainder of the section, we shall study JB^*-algebras which are Banach dual spaces, but are not necessarily the double dual of a JB^*-algebra. We shall follow the approach due to Sakai [69] for W^*-algebras, and require the following result from Banach space theory, which is given in [29], section 5·5.

THEOREM 3·3·7. Let X be a Banach space, S a convex subset of X', and $f \in X''$.

(i) S is $\tau(X',X)$ closed if and only if $S \cap r(X_1)$ is $\tau(X',X)$ closed for all $r \in \mathbb{R}^+$.

(ii) f is $\tau(X',X)$ continuous if and only if $f|_{(X')_1}$ is continuous in the relative $\tau(X',X)$ topology on $(X')_1$.

LEMMA 3·3·8. Let A be a unital JB^*-algebra which is the dual of a Banach space A_*. Then HerA and A^+ are $\tau(A,A_*)$ closed sets.

Proof. By Theorem 3·3·7 (i), as HerA and A^+ are wedges, it suffices to show that $A_1 \cap$ HerA and $A_1 \cap A^+$ are $\tau(A,A_*)$ closed sets. Let $\{x_\alpha\}$ be a net in $A_1 \cap$ HerA, and suppose $x_\alpha \to a + ib$ $\tau(A,A_*)$ where $a, b \in$ HerA, and $b \neq 0$. By considering $\{-x_\alpha\}$ if necessary, we may assume that there exists $\lambda \in \mathbb{R}^+ \setminus \{0\}$, such that $\lambda \in \sigma(b)$. For some large positive $n \in \mathbb{N}$ and each α,

$$\|x_\alpha + in\| \leq (1+n^2)^{\frac{1}{2}} < \lambda + n \leq \|b+n\| \leq \|a+i(b+n)\| .$$

Since $\{x_\alpha + in\}$ converges to $a + ib + in$, it follows by the $\tau(A,A_*)$ compactness of $(1+n^2)^{\frac{1}{2}}.A_1$ that $a + ib + in \in (1+n^2)^{\frac{1}{2}} A_1$. This contradicts the above inequality, and so $A_1 \cap$ HerA is $\tau(A,A_*)$ closed. As $A^+ \cap A_1$ is the homeomorphic image of $A_1 \cap$ HerA under the map $a \to \frac{1}{2}(a+1)$, $A^+ \cap A_1$ is also $\tau(A,A_*)$ closed.

THEOREM 3·3·9. Let A be an embedable JB^*-algebra. A has a unit if and only if there exists $x \in A_1$ such that, for all a in A,

$$a - 2\{x,x^*,a\} + \{x,\{x^*,a,x^*\},x\} = 0 . \qquad (\dagger)$$

Proof. If A has a unit, it is clear that 1 satisfies $(\dagger)$.
Conversely suppose $(\dagger)$ holds for some $x \in A_1$ and for all a in A. By Theorem 3·3·6, A'' is a unital JB^*-algebra, so by the separate $\tau(A'',A')$ continuity of the Arens product, and the $\tau(A'',A')$ density of A in A'', it follows that

$$a - 2\{x,x^*,a\} + \{x,\{x^*,a,x^*\},x\} = 0$$

for all a in A''. In particular

$$1 = 2\, x \circ x^* - \{x,(x^*)^2,x\} \in A .$$

A similar result, due to Sakai [69] and Miles [58] for B^*-algebras, is used to show that a B^*-algebra Y has a unit if and only if Y_1 has an extreme point. We were unable to show this also held for JB^*-algebras, but we note that (†) characterises the extreme points of the closed unit ball of a JC^*-algebra by Theorem 1·4·10, and the extreme points of the closed unit ball of a unital JB^*-algebra by recent work in [19]. We shall return to this problem in the next chapter.

COROLLARY 3·3·10. Let A be an embedable JB^*-algebra, and let B be the set of self-adjoint elements of A. If B_1 has an extreme point, b then, for all a in A, we have

$$a - 2b^2 \circ a + \{b^2, a, b^2\} = 0 .$$

Proof Let $d \in B \setminus \{0\}$. By Theorem 3·2·6, $P(b,d)$ is isometrically *-isomorphic to a JC^*-algebra. By Theorem 1·4·10, $b \pm \{(1-b^2)^{\frac{1}{2}}, (3\|d\|)^{-1}d , (1-b^2)^{\frac{1}{2}}\} \in B_1$, and hence, as b is an extreme point of B_1,

$$\{(1-b^2)^{\frac{1}{2}}, d, (1-b^2)^{\frac{1}{2}}\} = 0 .$$

Thus, $\{1-b^2, d, 1-b^2\} = 0$, and the general result follows easily from this.

By Corollary 3·3·10, if A is an embedable JB^*-algebra which has a predual A_* such that the set of self-adjoint elements is $\tau(A,A_*)$ closed, then A has a unit. Since, in order to follow the approach in [69] we must assume the set of self-adjoint elements $\tau(A,A_*)$ closed, there is therefore no loss in only dealing with unital JB^*-algebras.

For the remainder of this section, we shall keep to the following notation.

Notation A will denote a unital JB^*-algebra which is the dual of a Banach space A_*. S will denote the $\tau(A,A_*)$ continuous elements of $(A')^+$, and T the linear span of S. N will denote the set of all normal linear functionals on A, that is the set of all $f \in A'$ such that $f|_{HerA}$ is a normal real linear functional on $HerA$.

THEOREM 3·3·11. The involution is a $\tau(A,A_*)$ continuous map, and if E is a $\tau(A,A_*)$ closed real subspace of self-adjoint elements of A, then $E \oplus iE$ is $\tau(A,A_*)$ closed.

Proof. Let $\{x_\alpha\}$ be a net in A_1 such that $x_\alpha \to x$ $(\tau(A,A_*))$. Let $h_\alpha + ik_\alpha = x_\alpha$ be the standard decomposition of x_α. By the $\tau(A,A_*)$ compactness of A_1, there exist subnets, also denoted by h_α and k_α such that $h_\alpha \to h$ and $k_\alpha \to k$ for some h and k in A_1. By Lemma 3·3·8, h and k are self-adjoint. Thus

$$x = \lim x_\alpha = \lim(h_\alpha + ik_\alpha) = h + ik$$

and so

$$x^* = h - ik = \lim(h_\alpha - ik_\alpha) = \lim x_\alpha^* \quad .$$

It follows that, given $f \in A_*$, if $g \in A'$ is defined by

$$g(y) = (y^*(f))^*$$

for y in A, then $g|_{A_1}$ is $\tau(A,A_*)$ continuous. By Theorem 3·3·7, g is $\tau(A,A_*)$ continuous, and hence the involution is $\tau(A,A_*)$ continuous.

Finally, let $p : A \to HerA$ and $q : A \to iHerA$ be defined by $p(x) = \frac{1}{2}(x+x^*)$ and $q(x) = \frac{1}{2}(x-x^*)$ for x in A. As $*$ is $\tau(A,A_*)$ continuous, p and q are $\tau(A,A_*)$ continuous maps. Thus $E \oplus iE = p^{-1}(E) \cap q^{-1}(E)$ is $\tau(A,A_*)$ closed.

LEMMA 3·3·12. For any $a \in HerA$ such that $a \notin A^+$, there exists $\phi \in S$ such that $\phi(a) < 0$. In particular, if $b \in A$ is such that $\psi(b) = 0$ for all ψ in S, then $b = 0$.

Proof. As A^+ is a $\tau(A,A_*)$ closed convex set, there exists a $\tau(A,A_*)$ continuous real linear functional g on HerA such that $g(a) < \inf\{g(h) : h \in A^+\}$. As A^+ is a cone, $\inf\{g(h) : h \in A^+\} = 0$, and so $g(a) < 0 \leq g(h)$ for all $h \in A^+$. If we define $\phi \in A'$ by

$$\phi(x+iy) = g(x) + ig(y)$$

for x and y in HerA, then $\phi \in (A')^+$, and as the involution on A is $\tau(A,A_*)$ continuous, ϕ is $\tau(A,A_*)$ continuous. The remainder is clear.

THEOREM 3·3·13. HerA is a monotone complete JB-algebra with an invariant full set of normal states.

Proof. Let $\{a_\alpha\}$ be an increasing net in HerA which is bounded above by b. Without loss of generality, we may assume $a_\alpha \in A^+$ for all α, and so $\{a_\alpha\} \subseteq \|b\|A_1$. The compact Hausdorff topology $\tau(A,A_*)$ is stronger than the Hausdorff topology $\tau(A,T)$ on $\|b\|A_1$, and hence $\tau(A,A_*)$ and $\tau(A,T)$ agree on $\|b\|A_1$. For every $f \in S$, $\{f(a_\alpha)\}$ is a uniformly bounded increasing net in $\mathbb{R}$, and hence is Cauchy. By the $\tau(A,T)$ compactness of $\|b\|A_1$, $\{a_\alpha\}$ converges to an element $a \in \text{HerA} \cap \|b\|A_1$ in both the $\tau(A,A_*)$ and the $\tau(A,T)$ topologies. Moreover, by Lemma 3·3·12, $a = \sup\{a_\alpha\}$. Hence A is monotone complete.

By Lemma 3·3·12, $\{f : f \in S \text{ and } \|f|_{\text{HerA}}\| = 1\}$ is a full set of normal states, and hence it remains to show that if g is a normal state and $d \in$ HerA, then gU_d is a positive multiple of a normal state. This argument is due to Shultz [76].

As U_d is a positive operator, $gU_d \in (A')^+$, and so it suffices to show that if $\{b_\alpha\}$ is an increasing net in HerA with $b = \sup b_\alpha$, then $U_d(b_\alpha)$ is an increasing net with $U_d(b) = \sup U_d(b_\alpha)$. If d is invertible, $(U_d)^{-1} = U_{d^{-1}}$ and U_d are positive, so that U_d is an order automorphism of HerA, and hence the result follows. For any $d \in$ HerA, there exists $\lambda \in \mathbb{R}^+$ such that $\lambda + d$ and $\lambda - d$ are invertible. Thus $\sup U_{\lambda+d}(b_\alpha) = U_{\lambda+d}(b)$ and $\sup U_{\lambda-d}(b_\alpha) = U_{\lambda-d}(b)$. However, as $U_{\lambda+d} + U_{\lambda-d} = 2\lambda^2 + 2U_d$, we have

$$\begin{aligned}\lambda^2 b + U_d b &= \tfrac{1}{2}\sup((U_{\lambda+d} + U_{\lambda-d})(b_\alpha))\\ &= \sup(\lambda^2 b_\alpha + U_d(b_\alpha))\\ &= \lambda^2 b + \sup U_d b_\alpha\ .\end{aligned}$$

We shall now show that L_a is a $\tau(A,A_*)$ continuous operator. In fact, this follows from Theorem 3·3·1, and the above proof, but we take the following alternative approach.

THEOREM 3·3·14. Let e be a projection in A. Then the subalgebra $U_e A$ is $\tau(A,A_*)$ closed and the maps U_e and L_e are $\tau(A,A_*)$ continuous.

Proof $U_e A = \{x \in A : U_e x = x\}$ is a norm closed self-adjoint subalgebra with unit e and $U_e(A^+ \cap A_1) = \{a \in A^+ \cap A_1 : a \leq e\}$. Hence, if $\{x_\alpha\}$ is a net in $U_e(A_1 \cap \text{HerA})$ converging to x, $(\tau(A,A_*))$, then $x \in A^+ \cap A_1$, and $e - x = \lim(e - x_\alpha) \in A^+$, as A^+ is $\tau(A,A_*)$ closed. Hence, $x \in U_e(A^+ \cap A_1)$ and so $U_e(A^+ \cap A_1)$ is $\tau(A,A_*)$ closed. As $U_e(A_1 \cap \text{HerA}) = U_e(A^+ \cap A_1) - U_e(A^+ \cap A_1)$ the $\tau(A,A_*)$ compactness of $U_e(A^+ \cap A_1)$ implies that $U_e(A_1 \cap \text{HerA})$ is a $\tau(A,A_*)$ compact set. In addition, as $1 - e$ is also a projection, $U_{1-e}(A_1 \cap \text{HerA})$ is a $\tau(A,A_*)$ compact set.

Next, we show that if $\{a_\alpha\}$ is a net in $A_1 \cap \mathrm{Her}A$ and $\{e,a_\alpha,1-e\} \to a$ $(\tau(A,A_*))$, then $U_e a = U_{1-e} a = 0$. For any $s \in \mathbb{R}^+$, identifying $P(1,e,a_\alpha)$ as a JC*-algebra, we have

$$\begin{aligned}
\|\{e,a_\alpha,1-e\} + se\| &\leq \tfrac{1}{2}(\|ea_\alpha(1-e) + se\| + \|(1-e)a_\alpha e + se\|) \\
&= \|(ea_\alpha(1-e) + se)(1-e)a_\alpha e + se)\|^{\frac{1}{2}} \\
&= \|ea_\alpha(1-e)a_\alpha e + s^2 e\|^{\frac{1}{2}} \\
&\leq \|a_\alpha(1-e)a_\alpha + s^2\|^{\frac{1}{2}} \\
&\leq (1+s^2)^{\frac{1}{2}} .
\end{aligned}$$

As $a_\alpha \in \mathrm{Her}A$ for all α, $a \in \mathrm{Her}A$ and so $U_e(a) \in \mathrm{Her}A$. If $U_e(a) \neq 0$, $U_e(a)$ is a non-zero self-adjoint element of the unital JB*-algebra $U_e A$, and so there is a non-zero λ in the spectrum of $U_e(a)$ in $U_e A$. By considering $\{-a_\alpha\}$ if necessary, we may assume $\lambda \in \mathbb{R}^+$. Thus

$$\lambda + s \leq \|U_e a + se\| \leq \|a + se\| .$$

So, for large enough s, $\|a + se\| > (1+s^2)^{\frac{1}{2}}$, which contradicts the $\tau(A,A_*)$ compactness of $(1+s^2)^{\frac{1}{2}}A_1$. Hence $U_e(a) = 0$, and by symmetry, $U_{1-e}(a) = 0$. As, for all y in A, we have $y = U_e(y) + U_{1-e}(y) + 2\{e,y,1-e\}$, it follows that the $\tau(A,A_*)$ closure of $\{e, A_1 \cap \mathrm{Her}A, 1-e\}$ is contained in $A_1 \cap \mathrm{Her}A \cap \{e,A,1-e\}$.

Now, let $\{x_\alpha\}$ be a net in A_1 and suppose that $x_\alpha \to x$ $(\tau(A,A_*))$. Let $x_\alpha = h_\alpha + ik_\alpha$ be the standard decomposition of x_α and $x = h + ik$ be the standard decomposition of x. As the involution is $\tau(A,A_*)$ continuous, $h_\alpha \to h$ and $k_\alpha \to k$. Moreover $\{h_\alpha\}$ and $\{k_\alpha\}$ are contained in $A_1 \cap \mathrm{Her}A$. Let $a_\alpha = U_e h_\alpha$, $b_\alpha = U_{1-e} h_\alpha$, and $c_\alpha = 2\{e,h_\alpha,1-e\}$. By the compactness of A_1, there are subnets, also denoted by $\{a_\alpha\}$, $\{b_\alpha\}$, and $\{c_\alpha\}$ and $a, b, c \in A_1 \cap \mathrm{Her}A$ such that $a_\alpha \to a$, $b_\alpha \to b$ and $c_\alpha \to c$. By the previous argument moreover,

$a \in U_eA$, $b \in U_{1-e}A$ and $c \in \{e,A,1-e\}$. So

$$U_e(h) = U_e(a+b+c) = a = \lim a_\alpha = \lim U_e(h_\alpha) .$$

Similarly, $U_e k = \lim U_e k_\alpha$, and so

$$U_e(x) = \lim U_e(x_\alpha) .$$

Given $f \in A_*$, define $g \in A'$ by $g(y) = (U_e(y))(f)$ for y in A . By the previous paragraph, $g|_{A_1}$ is $\tau(A,A_*)$ continuous. Hence, by Theorem 3·3·7, g is $\tau(A.A_*)$ continuous. So U_e is $\tau(A,A_*)$ continuous. It follows immediately that $L_e + \frac{1}{2}(I+U_e-U_{1-e})$ is $\tau(A,A_*)$ continuous.

Finally if $\{y_\alpha\}$ is a net in U_eA such that $y_\alpha \to y$ $(\tau(A,A_*))$, then $U_e y = \lim U_e y_\alpha = \lim y_\alpha = y$, and so U_eA is $\tau(A,A_*)$ closed.

COROLLARY 3·3·15. For all a in A, L_a and U_a are $\tau(A,A_*)$ continuous operators.

<u>Proof</u>. It suffices to prove that L_h is $\tau(A,A_*)$ continuous for all $h \in \mathrm{Her}A$. If $h \in \mathrm{Her}A$, then, by Theorems 3·3·13 and 1·3·7, for any $\eta \in \mathbb{R}^+ \setminus \{0\}$, there exists a finite family of projections $\{e_j\}$ in A and a finite set $\{\lambda_j\}$ in $\mathbb{R}$ such that $\|h - \sum_{j=1}^{n} \lambda_j e_j\| < \eta$.

Let $\{x_\alpha\}$ be a net in A_1 such that $x_\alpha \to x$ $(\tau(A,A_*))$. As $\frac{1}{2}(x_\alpha - x) \to 0$, it suffices to consider the case when $x = 0$. Then, given $f \in A_*$, we have

$$|(h o x_\alpha)(f)| \leq |((h-\sum_{j=1}^{n}\lambda_j e_j) o x_\alpha)(f)| + |((\sum_{j=1}^{n}\lambda_j e_j) o x_\alpha)(f)|$$

$$\leq \eta\|f\| + \sum_{j=1}^{n}|\lambda_j|\,|(L_{e_j}(x_\alpha))(f)| \quad .$$

Thus $\overline{\lim}\, |(h o x_\alpha)(f)| \leq \eta\|f\|$, and so $\lim (h o x_\alpha)(f) = 0$.

Hence, if $g \in A'$ is defined by $g(x) = (h o x)(f)$ for all x in A , $g|_{A_1}$ is $\tau(A,A_*)$ continuous, and so g is $\tau(A,A_*)$ continuous.

Thus, L_h is a $\tau(A,A_*)$ continuous operator.

We remark that in [61], von Neumann essentially proved a "converse" to Corollary 3·3·15 as follows: if B is a JB-algebra with a predual B_*, such that for all a in B, L_a is a $\tau(B,B_*)$ continuous operator, then B has a unit.

It is clear that we have only started the theory of unital JB^*-algebras which are Banach dual spaces, along the lines suggested in [69]. However, in the next section we shall be more concerned with generalising results of Topping [84].

3.4 The centre and the quadratic ideals of a JB^*-algebra.

DEFINITION. Let A be an embedable JB^*-algebra. We denote by $Z(A)$ the set of self-adjoint elements of A which operator commute with all self-adjoint elements of A. The centre of A, $\mathcal{C}(A)$ is defined by $\mathcal{C}(A) = Z(A) \oplus iZ(A)$.

At first sight, it is not clear that the centre of an embedable JB^*-algebra is a Jordan subalgebra. In essence von Neumann in [61] showed that if A is a unital JB-algebra which is a Banach dual space, then $Z(A)$ is a Jordan subalgebra of A. We shall use the following characterisation of operator commutativity, which extends results of Topping [84] and Alfsen, Shultz and Størmer [5], to derive a more general result.

THEOREM 3·4·1. Let A be a unital JB^*-algebra, and let a and b be self-adjoint elements of A. The following are equivalent :

(i) a operator commutes with b in A ,

(ii) a operator commutes with b in A" ,

(iii) a operator commutes with all e in the spectral family of b in A" ,

(iv) $L_e a = U_e a$ for all e in the spectral family of b in A",

(v) whenever J is a JC^*-algebra, K is a closed self-adjoint subalgebra of A containing a and b , and $F : K \to J$ a Jordan *-homomorphism, then $F(a)F(b) = F(b)F(a)$,

(vi) $b^2 o\ a = \{b,a,b\}$.

Proof . We first note by Theorems 3·3·4, 3·3·13, and 1·3·7, that each self-adjoint x in A has a spectral family in Her(P(1,x))" .

(i) ⇒ (vi). If L_a commutes with L_b , we have, in particular, that

$$a\ o(bob) = b\ o(aob) .$$

As $(L_b)^2 = \frac{1}{2}(L_{b^2} + U_b)$, it follows that $b^2 o\ a = \{b,a,b\}$.

(vi) ⇒ (v) . Let J be a JC^*-algebra, K a closed self-adjoint subalgebra of A containing a and b , and $F : J \to K$ a Jordan *-homomorphism. Then F(a) and F(b) are self-adjoint, and

$$\tfrac{1}{2}((F(b)^2F(a) + F(a)(F(b)^2) = F(b)F(a)F(b)$$

and so $$F(b)(F(b)F(a) - F(a)F(b)) = (F(b)F(a) - F(a)F(b))F(b) .$$

By the Kleinecke-Shirokov theorem, see for example [14] Proposition 18·13 , $F(a)F(b) - F(b)F(a)$ is quasinilpotent. However, as F(a) and F(b) are self-adjoint, $i(F(a)F(b) - F(b)F(a))$ is self-adjoint. Hence $F(a)F(b) = F(b)F(a)$, as required.

(v) ⇒ (iv). Let J = P(1,a,b). By Theorem 2·3·5, there exists an isometrical *-isomorphism F of J onto a JC^*-algebra K . As $F(a)F(b) = F(b)F(a)$, it follows that K is a commutative C^*-algebra. Thus K" is a commutative B^*-algebra under the Arens product. $F'' : J'' \to K''$ is an isometrical *-isomorphism, and, as the spectral family of b lies in $\mathrm{Her}(P(1,b))''$, which we can regard as a subalgebra of J'', it follows that

$$F''(e)F''(a) = F''(a)F''(e)$$

for all e in the spectral family of B . Hence

$$e \circ a = \{e,a,e\}$$

for all e in the spectral family of b.

(iv) ⇒ (iii). This follows by Theorem 1·1·12.

(iii) ⇒ (ii). Let $\eta \in \mathbb{R}^+ \setminus \{0\}$. By Theorem 1·3·7, there is a finite set of projections $e_1,\ldots,e_n$ in the spectral family of b , and a finite set $\lambda_1,\ldots,\lambda_n$ of real numbers such that

$$\|b - \sum_{j=1}^{n} \lambda_j e_j\| < \eta .$$

Hence $\|L_b - \sum_{j=1}^{n} \lambda_j L_{e_j}\| < \eta$, and so, as a operator commutes with every e_j, it follows that

$$\begin{aligned} \|L_a L_b - L_b L_a\| &\leq 2\|L_a\|\, \|L_b - \sum_{j=1}^{n} \lambda_j L_{e_j}\| \\ &= 2\|L_a\|\, \eta . \end{aligned}$$

Thus a operator commutes with b .

(ii) ⇒ (i) . This is trivial.

We note that it may happen that the spectral family of b is in A. For example, if M is a monotone complete unital JB-algebra, with a full invariant set of normal states, then $Z(M \oplus iM)$ coincides with $Z(M)$ defined after Theorem 1·3·7.

THEOREM 3·4·2. If A is an embedable JB^*-algebra, then $\mathfrak{C}(A)$ is a commutative B^*-algebra.

<u>Proof</u>. It is clear that $\mathfrak{C}(A)$ is a closed complex subspace of A . To show that $\mathfrak{C}(A)$ is a Jordan algebra, it suffices to show that

$Z(A)$ is a Jordan algebra. Let $a \in Z(A)$, and let b be a self-adjoint element of A . Let J be the JB*-algebra obtained by adjoining a unit to A , and let $c + \mu \in J$, where $c \in A$ and $\mu \in \mathbb{C}$.

$$a\, o(bo(c+\mu)) = b\, o(ao(c+\mu)) ,$$

and so by Theorem 3·4·1, it follows that $\{a,b,a\} = a^2 ob$. Thus

$$\begin{aligned}\{a^2,b,a^2\} &= \{a,\{a,b,a\},a\} \\ &= \{a,a^2ob,a\} \\ &= a^2o(a^2ob) \\ &= \tfrac{1}{2}(a^4ob+\{a^2,b,a^2\}) .\end{aligned}$$

Hence $\{a^2,b,a^2\} = (a^2)^2 o\, b$, and so it again follows by Theorem 3·4·1 that $a^2 \in Z(A)$. By the definition of operator commutativity, the Jordan product is associative on $Z(A)$, and hence the result follows by Lemma 2·3·1.

From the theory of partially ordered vector spaces with order unit, a more general definition of centre was given by Wils in [91]. From the next Theorem, we shall show that this coincides with $Z(A)$ for a unital JB*-algebra. The possibility of this result was pointed out to me by Professor Wright. The proof is a slight modification of the analogous result for B*-algebras given by Andersen [6]. First we give the following definition due to Wils [91].

DEFINITION. Let A be a unital JB*-algebra. A linear self-adjoint map $T : A \to A$ is called order bounded if there exists $\lambda \in \mathbb{R}^+$ such that

$$-\lambda a \leq Ta \leq \lambda a$$

for all $a \in A^+$. (We note that an order bounded map T is continuous and $\|T\| \leq 4\lambda$.)

THEOREM 3·4·3. Let A be a unital JB^*-algebra.

(i) If $a \in Z(A)$, then L_a is an order bounded map.

(ii) If T is an order bounded map on A, then $T(1) \in Z(A)$.

Proof. (i) Let $a \in Z(A)$, and let $b \in A^+$. By Theorem 2·3·5, we can regard $P(1,a,b)$ as a unital JC^*-algebra. By Theorem 3·4·1, $ab = ba$, and so $P(1,a,b)$ is a commutative C^*-algebra. Hence,

$$-\|a\|\, b \leq L_a b \leq \|a\|\, b\ ,$$

and so L_a is order bounded.

(ii) By replacing T by $(2\lambda)^{-1}(\lambda I+T)$ if necessary, we may assume

$$0 \leq Ta \leq a$$

for all $a \in A^+$. Let $q = T|_{HerA}$, so that $q : HerA \to HerA$ is a continuous real linear map. Let B denote the envelopping algebra of HerA in (HerA)" constructed in [5] Section 3. By an analogue of the Kaplansky density theorem, [5] Proposition 3·9, if $a \in B^2$ there exists a net $\{a_\alpha\}$ in HerA such that $a_\alpha \in \|a\|^{\frac{1}{2}}A_1$ and $a_\alpha^2 \to a$ $\tau((HerA)",(HerA)')$. As $q":(HerA)" \to (HerA)"$ is $\tau((HerA)",(HerA)')$ continuous, it follows that

$$0 \leq q"(a) \leq a\ ,$$

by Corollary 1·3·14 and [76] Theorem 2·3.

Now let p be an idempotent in B. As $0 \leq q"(p) \leq p$ and (HerA)" is a unital JB-algebra, $q"(p) = U_p(q"(p))$. Hence

$$\begin{aligned} q"(1)\circ p &= (q"(p))\circ p + (q"(1-p))\circ p \\ &= q"(p) + L_p U_{1-p}(q"(1-p)) \\ &= q"(p)\ . \end{aligned}$$

Also
$$\begin{aligned} U_p(q"(1)) &= U_p(q"(p)) + U_p(q"(1-p)) \\ &= q"(p)\ , \end{aligned}$$

and so $L_p(q''(1)) = U_p(q''(1))$. As B is a monotone complete unital JB-algebra with a full set of normal states, B contains the spectral family of every element of B . Hence, by Theorems 2·4·7 and 3·4·1 , $q''(1) = q(1)$ operator commutes with every element of B. In particular, $q(1) = T(1) \varepsilon Z(A)$.

In [91], Wils defined the centre of a partially ordered vector space with order unit as $\{T(1):T$ is an order bounded map$\}$, Hence, we see that for unital JB*-algebras the two definitions of centre coincide.

We now turn to a generalisation of some of Topping's work on quadratic ideals and annihilators. In the previous section, we used the symbol $^\perp$ to denote the Banach space annihilator of a set. This symbol will not be used again in that context, so we shall use $^\perp$ with a different meaning in this section.

DEFINITION (i) Let A be a JB*-algebra, and let B be the set of self-adjoint elements of A . An absolute order ideal I is a self-adjoint complex linear subspace of A such that if $h \varepsilon B \cap I$, then $|h| = 2h^+ - h \varepsilon B \cap I$, and if $a \varepsilon I$ and $b \varepsilon B$ are such that $0 \leqslant b \leqslant a$, then $b \varepsilon I$.

(ii) A quadratic ideal of a Jordan algebra J is a linear subspace I of J such that, whenever $a \varepsilon I$ and $b \varepsilon J$, we have $\{a,b,a\} \varepsilon J$.

In view of Theorem 2·4·12, it might be expected that every quadratic ideal of a unital JB*-algebra would be self-adjoint. However, this is false even for von Neumann algebras.

Example Let $\mathcal{H}$ be the complex Hilbert space of all sequences $\{x_n\}$ of complex numbers such that $\sum_{n=0}^{\infty} |x_n|^2 < \infty$, with the usual inner product. Let $T \in B(\mathcal{H})$ be the unilateral shift, defined by $T(\{x_n\}) = \{y_n\}$, where
$$y_n = \begin{cases} 0 & \text{if } n = 0. \\ x_{n-1} & \text{if } n > 0 . \end{cases}$$

It is well known that $T^*(\{x_n\}) = \{z_n\}$ where $z_n = x_{n+1}$ for $n \geq 0$, and hence T is a partial isometry such that $T(T^*)^2 \neq (T^*)^2T$.

If $J = U_T(B(\mathcal{H}))$, then J is a weak operator closed quadratic ideal of $B(\mathcal{H})$, and $T \in J$. Suppose that $T^* \in J$. Then there exists $A \in B(\mathcal{H})$ such that $T^* = TAT$. Then, as $T = TT^*T$, we have
$$TT^*T^* = TT^*TAT = TAT = TATT^*T = T^*T^*T ,$$
which is a contradiction. Hence $T^* \notin J$, so that J is not self-adjoint.

THEOREM 3·4·4. Let A be a unital JB^*-algebra.

(i) If J is a self-adjoint complex linear subspace of A, then J is a quadratic ideal of A if and only if $J \cap \mathrm{Her}A$ is a quadratic ideal of $\mathrm{Her}A$.

(ii) If I is a closed self-adjoint complex linear subspace of A, then I is an absolute order ideal if and only if I is a quadratic ideal.

Proof.(i) Clearly, if J is a quadratic ideal of A, $J \cap \mathrm{Her}A$ is a quadratic ideal of $\mathrm{Her}A$. Conversely, let $x \in J$, and let $x = h + ik$ be the standard decomposition of x. Then $h, k \in J$, and so, for all y in $\mathrm{Her}A$,
$$\{h,y,k\} = \tfrac{1}{2}(U_{h+k} - U_h - U_k)(y) \in J .$$
Hence, for all y in $\mathrm{Her}A$,
$$U_xy = \{h,y,h\} - \{k,y,k\} + 2i\{h,y,k\} \in J .$$

It now follows easily that J is a quadratic ideal of A .

(ii) Suppose that I is an absolute order ideal, and let $a \in I \cap \mathrm{Her}A$ and $b \in \mathrm{Her}A$. By Theorem 2·3·5, we may regard $P(1,a,b)$ as a unital JC^*-algebra and $I \cap P(1,a,b)$ is an absolute order ideal of $P(1,a,b)$. By [84], Theorem 2, $\{a,b,a\} \in I \cap P(1,a,b)$, and so $I \cap \mathrm{Her}A$ is a quadratic ideal. Hence by (i), I is a quadratic ideal of A .

Conversely suppose I is a quadratic ideal, and let $b \in I \cap \mathrm{Her}A$. As $b^2 = U_b(1)$, it follows that $P(1,b) \subseteq I$, and hence $|b| \in I$. If in addition $a \in \mathrm{Her}A$ is such that $0 \leq a \leq b$, by Theorem 2·3·5, we may regard $P(1,a,b)$ as a unital JC^*-algebra, and $I \cap P(1,a,b) \cap \mathrm{Her}A$ is a closed quadratic ideal of $P(1,a,b) \cap \mathrm{Her}A$. Thus by [84] Theorem 2, $a \in I$ and so I is an absolute order ideal.

DEFINITION. Let A be a unital JB^*-algebra, and let M be a subset of $\mathrm{Her}A$. The annihilator of M , denoted by $M^{\perp}$, is defined by

$M^{\perp} = \{a \in \mathrm{Her}A : L_a$ commutes with L_b and $\{b,a,b\} = 0$ for all b in $M\}$.

THEOREM 3·4·5. Let A be a unital JB^*-algebra, and let M be a subset of $\mathrm{Her}A$.

(i) Let $b \in M$ and $a \in \mathrm{Her}A$. $a \in M^{\perp}$ if and only if, whenever J is a closed self-adjoint subalgebra of A containing 1, a, and b, K is a JC^*-algebra and $F : J \to K$ a Jordan *-homomorphism, then $F(a)F(b) = 0$.

(ii) $M^{\perp}$ is a closed subalgebra and quadratic ideal of $\mathrm{Her}A$.

Proof . (i) Suppose $a \in M^{\perp}$, J is a closed self-adjoint subalgebra of A containing 1, a and b , K is a JC^*-algebra, and $F : J \to K$

a Jordan *-homomorphism. By Theorem 3·4·1, $F(a)F(b) = F(b)F(a)$, and so

$$(F(b)F(a))(F(b)F(a))^* = F(b)F(a)F(b)F(a) = 0 \ .$$

Hence $F(b)F(a) = F(a)F(b) = 0$.

Conversely, if $J = P(1,a,b)$, by Theorem 2·3·5, there is an isometrical *-isomorphism F of J onto a unital JC*-algebra. Then by assumption, $(F(b))^2 \circ F(a) = \{F(b),F(a),F(b)\} = 0$. As F is an isometrical *-isomorphism, we conclude that $b^2 \circ a = \{b,a,b\} = 0$. Hence, by Theorem 3·4·1, $a \in M^\perp$.

(ii) It is clear that $M^\perp$ is a closed real linear subspace of HerA . Let $a \in M^\perp$, and let $b \in M$. By Theorem 2·3·5, we may regard $P(1,a,b)$ as a JC*-algebra, and so by (i), it follows that $ab = 0$. Hence

$$b^2 \circ a^2 = \{b,a^2,b\} = 0 \ ,$$

and so, by Theorem 3·4·1, $a^2 \in M^\perp$. Thus $M^\perp$ is a Jordan subalgebra of HerA , and in particular, $|a| \in M$.

Now suppose $a \in M^\perp$, and that $a \geq d \geq 0$, where $d \in$ HerA . Then

$$0 = U_b a \geq U_b d \geq 0$$

for all b in M . Let $b \in M$. By Theorem 2·3·5, we may regard $P(1,b,d)$ as a JC*-algebra, and so, as $0 = (bd^{\frac{1}{2}})(bd^{\frac{1}{2}})^*$, it follows that $bd = 0$. Hence

$$b^2 \circ d = \{b,d,b\} = 0 \ ,$$

and so $d \in M^\perp$. Thus, by Theorem 3·4·4, $M^\perp$ is a closed quadratic ideal of HerA .

THEOREM 3·4·6. Let A be a unital JB*-algebra, which has a predual A_* , and let I be a self-adjoint complex linear subspace of A. Then the following are equivalent ;

(i) I is a $\tau(A,A_*)$ closed quadratic ideal of A ,

(ii) I is a $\tau(A,A_*)$ closed absolute order ideal of A ,

(iii) $I = M^{\perp} \oplus iM^{\perp}$ for some set $M \subseteq \text{Her}A$,

(iv) $I = U_e A$ for some projection $e \in A$.

Proof. (i)<=>(ii) follows by Theorem 3·4·4.

(i) ⇒ (iv) We have already noted that if I is a quadratic ideal, then I is a Jordan subalgebra. Hence, by Lemma 3·3·8, and Corollary 3·3·10, there is a projection e in A such that e is the unit for A . As I is a quadratic ideal,

$$U_e A \subseteq I = U_e I \subseteq U_e A \quad ,$$

and so $I = U_e A$.

(iv) ⇒ (iii) . Let $M = \{1 - e\}$. By Lemma 1·1·11 and Theorem 1·1·12 , $U_e(\text{Her}A) \subseteq M^{\perp}$. Conversely, if $x \in M^{\perp}$, $L_{1-e}(x) = U_{1-e}(x) = 0$, and so $x = x \circ e$. Hence $U_e x = x$, and so $x \in U_e(\text{Her}A)$. Thus $M^{\perp} \subseteq U_e(\text{Her}A)$. So $M^{\perp} = U_e(\text{Her}A)$, and hence $M^{\perp} \oplus iM^{\perp} = U_e A$.

(iii) ⇒ (i) . Let $\{a_\alpha\}$ be a net in $M^{\perp}$ converging to $a \in A$ $(\tau(A,A_*))$. As $\text{Her}A$ is $\tau(A,A_*)$ closed, $a \in \text{Her}A$. Given $b \in M$, we have

$$b^2 \circ a = \lim b^2 \circ a_\alpha = \lim\{b,a_\alpha,b\} = \{b,a,b\}$$

and

$$\{b,a,b\} = \lim\{b,a_\alpha,b\} = 0$$

by Corollary 3·3·15 and Theorem 3·4·1. Hence $a \in M^{\perp}$ by Theorem 3·4·1, and so $M^{\perp}$ is $\tau(A,A_*)$ closed. By Theorem 3·3·11, $M^{\perp} \oplus iM^{\perp}$ is $\tau(A,A_*)$ closed. I is a quadratic ideal by Theorem 3·4·5 (ii) and Theorem 3·4·4(i) .

Part of the above Theorem, along with the following Corollary was obtained by Edwards [30].

COROLLARY 3·4·7. Let A be a unital JB*-algebra which has a predual

A_* . I is a $\tau(A,A_*)$ closed Jordan ideal if and only if $I = U_p A$ for some central projection p .

Proof. Suppose $I = U_p A$ where $p \varepsilon Z(A)$. By Theorem 3·4·6, I is a $\tau(A,A_*)$ closed quadratic ideal. Moreover, if $a \varepsilon U_p A$, and $b \varepsilon A$, then

$$b \circ a = L_b a = L_b L_p a = L_p L_b a = U_p(a \circ b)$$

by Theorem 1·1·12, and so I is an ideal.

Conversely, suppose that I is a $\tau(A,A_*)$ closed ideal. As an ideal is a self-adjoint quadratic ideal, it follows from Theorem 3·4·6, that $I = U_p A$ for some projection $p \varepsilon I$. To complete the proof, we have to show that $p \varepsilon Z(A)$, and, by Theorems 3·4·1 and 3·3·13, it suffices to show that p operator commutes with all projections, or with all $s \varepsilon$ HerA such that $s^2 = 1$. The following argument is due to Shultz , [76], Lemma 2·1.

Let $s \varepsilon$ HerA such that $s^2 = 1$. As I is an ideal, $U_s p \varepsilon I$. Also

$$(U_s p)^2 = \{s,p,s\}^2 = \{s,\{p,s^2,p\},s\} = \{s,p,s\} = U_s p \ ,$$

so that $U_s p \leq p$. By the positivity of U_s and the fact that $(U_s)^2 = I$, it follows that

$$p = (U_s)^2 p \leq U_s p \leq p \ ,$$

so that $U_s p = p$. Hence, by Theorem 3·4·1, s operator commutes with p , as required.

We remark that, under the hypothesis of Corollary 3·4·7, a $\tau(A,A_*)$ closed ideal of A is complemented. In essence, von Neumann obtained this in [61].

CHAPTER 4

The main result of this Chapter is Theorem 4·3·6, which states that if F is a linear isometry of an embedable JB*-algebra A onto itself, then $F\{x,x^*,x\} = \{F(x),(F(x))^*, F(x)\}$ for all x in A. We prove this by initially tackling the simpler problem when A is unital and $F(1) = 1$. This simpler result was conjectured by Kaplansky, and we prove it in section 1. An alternative proof appears in [95]. In section 2, we apply this result to find algebraic characterisations of the Hermitian elements of $B(A)$. As a consequence, we produce a large natural class of Hermitian operators whose squares are not Hermitian.

In section 3, we show that the open unit ball of a unital JB*-algebra is a bounded symmetric homogeneous domain, and, from this, deduce our main result. Finally, in section 4, we indicate how the theory of JB*-algebras is related to the study of certain bounded symmetric homogeneous domains. This is mainly a survey of the work of Kaup and his colleagues, and is not my original work.

4·1. Kadison's Theorem for JB^*-algebras.

This section is concerned with showing that a linear isometry F of a unital JB^*-algebra A onto a second unital JB^*-algebra B such that $F(1) = 1$, is a Jordan *-homomorphism. This was conjectured by Kaplansky, and extends a well known theorem of Kadison [46]. Our proof will follow the original proof of Kadison for B^*-algebras; a different approach was taken in [95].

We start off with results on the extreme points of the unit ball of a unital JB^*-algebra.

DEFINITION. Let A be a unital JB^*-algebra. $u \in A$ is called unitary if u is invertible and $u^{-1} = u^*$. $x \in A$ is called a partial isometry if $\{x,x^*,x\} = x$.

LEMMA 4·1·1. Let A be a unital JB^*-algebra.

(i) 1 is an extreme point of A_1.

(ii) If u is a unitary element of A, then u is a partial isometry.

(iii) If x is a non-zero partial isometry of A, then $\|x\| = 1$.

(iv) If $u \in A_1$, then u is unitary if and only if $u \circ u^* = 1$.

(v) If u is unitary, then u is an extreme point of A_1.

<u>Proof</u> (i). Suppose that $1 = \frac{1}{2}(c+d)$, where c and d are in A_1. Then if $a = \frac{1}{2}(c+c^*)$, and $b = \frac{1}{2}(d+d^*)$, it follows that a and b are in A_1, and $1 = \frac{1}{2}(a+b)$. Thus $a = 2 - b$, and so $P(1,a,b)$ is an associative subalgebra of A, and thus is a commutative B^*-algebra. Hence, by [69], Proposition 1·6·6, $a = b = 1$. Applying the same technique again, it follows that $P(1,c,c^*)$ is a commutative

B^*-algebra, with $1 = \frac{1}{2}(c+c^*)$, so again by [69], Proposition 1·6·6, $c = c^* = 1$. Hence $d = d^* = 1$, and so 1 is an extreme point of A_1.

(ii) This follows by Theorem 1·1·8.

(iii) This is clear from the definition of a JB^*-algebra.

(iv) If u is unitary, then $u \circ u^* = 1$. Conversely, suppose $u \circ u^* = 1$. If we regard $P(1,u,u^*)$ as a JC^*-algebra, we have

$$1 = \tfrac{1}{2}(uu^*+u^*u) .$$

Hence, by [69] Proposition 1·6·6, or (i), $uu^* = u^*u = 1$. Hence u is unitary.

(v) If u is unitary, by (ii) and (iii), $\|u\| = 1$. Suppose that $u = \frac{1}{2}(x+y)$ where $x, y \in A_1$. Then

$$1 = u \circ u^* = \tfrac{1}{4}(x\circ x^*+ x^*\circ y + x\circ y^*+ y\circ y^*) .$$

As $x\circ x^*$, $x^*\circ y$, $x\circ y^*$ and $y\circ y^*$ are all in A_1 , by (i), we have

$$x\circ x^* = x\circ y^* = y\circ x^* = y\circ y^* = 1 .$$

By (iv), x is unitary, and so by Theorem 1·1·10, $Q(1,x,x^*,u,u^*)$ is special. By Corollary 2·4·10, we conclude that $P(1,x,x^*,u,u^*)$ is $*$-isometrically isomorphic to a JC^*-algebra. By Theorem 1·4·10, u is an extreme point of $(P(1,u,u^*,x,x^*))_1$, and so $u = x = y$. Hence u is an extreme point of A_1 .

LEMMA 4·1·2 Let A be a unital JB^*-algebra.

(i). If x is an extreme point of A_1 , then x is a partial isometry.

(ii) If x is an invertible partial isometry, then x is unitary.

(iii) If $a, b \in A^+$ and $a + ib$ is unitary, then $b = (1-a^2)^{\frac{1}{2}}$.

Proof. (i) x is also an extreme point of $(P(1,x,x^*))_1$, and so the result follows by Theorems 2·3·5 and 1·4·10.

(ii) By Theorem 1·1·8, U_x is invertible, and $x^{-1} = (U_x)^{-1} x$. Thus

$$x^* = U_x^{-1} U_x x^* = U_x^{-1} x = x^{-1} .$$

(iii) As $1 = (a+ib)\circ(a-ib) = a^2 + b^2$, we have

$$1 = \|a^2 + b^2\| \geq \|a^2\| ,$$

and so $(1-a^2)^{\frac{1}{2}}$ is well defined. Moreover, as b is positive, and positive square roots are unique,

$$b = (b^2)^{\frac{1}{2}} = (1-a^2)^{\frac{1}{2}} .$$

THEOREM 4·1·3. Let A and B be unital JB*-algebras, and let $G : A \to B$ be a linear isometry of A onto B with $G(1) = 1$. If $a \in A_1 \cap A^+ \cap \mathrm{Inv}A$, and $x = a + i(1-a^2)^{\frac{1}{2}}$, then $G(x)$ is unitary.

Proof. As $a \in A_1 \cap A^+$, x is a well defined element of $P(a)$, and an easy calculation shows that x is unitary. By Lemma 4·1·1, x is an extreme point of A_1 , and so, as G is an isometry, $G(x)$ is an extreme point of B_1 .

Let $\alpha = \inf\{\lambda : \lambda \in \sigma(a)\}$, and choose $n \in \mathbb{N}$ such that $2n\alpha > 1$. Then

$$\begin{aligned}\|G(x) - n\| &= \|x - n\| \\ &= \|a - n + i(1-a^2)^{\frac{1}{2}}\| \\ &= \sup\{(|\lambda-n|^2 + |1-\lambda^2|)^{\frac{1}{2}} : \lambda \in \sigma(a)\} \\ &= \sup\{|n^2-2\lambda n+1|^{\frac{1}{2}} : \lambda \in \sigma(a)\} \\ &= |n^2-2\alpha n+1|^{\frac{1}{2}} \\ &< n .\end{aligned}$$

Thus $\|n^{-1}G(x) - 1\| < 1$, and so $G(x)$ is invertible, by Lemma 1·2·2. Hence $G(x)$ is unitary by Lemma 4·1·2.

THEOREM 4·1·4. Let A and B be unital JB*-algebras, and let $G : A \to B$ be a linear isometry of A onto B with $G(1) = 1$. Then G is a Jordan *-homomorphism.

Proof. As G is an isometry of A onto B with $G(1) = 1$, G' is an isometry of B' onto A' which maps $D(B,1)$ onto $D(A,1)$. Hence, if $x \in \mathrm{Her}A$, and $f \in D(B,1)$, then $f(G(x)) = (G'(f))(x) \in \mathbb{R}$, and so $G(x) \in \mathrm{Her}B$. By symmetry, G^{-1} maps $\mathrm{Her}B$ into $\mathrm{Her}A$, and so G maps $\mathrm{Her}A$ onto $\mathrm{Her}B$. Hence, by Theorem 2·3·4, $G(x^*) = (G(x))^*$ for all x in A.

To show that G is a Jordan homomorphism, it suffices to show that $G(c^2) = (G(c))^2$ for all c in $\mathrm{Her}A$, as the remainder of the argument is similar to that in [46]. If $b \in A^+ \cap A_1$, then $G(b)$ is self-adjoint and

$$\|G(b) - 1\| = \|b - 1\| < 1 ,$$

so that $G(b) \in A^+ \cap A_1$. Hence G preserves the order structure.

Now suppose $a \in A^+ \cap A_1 \cap \mathrm{Inv}A$. By Theorem 4·1·3, $G(a+i(1-a^2)^{\frac{1}{2}})$ is unitary. As G preserves the order, $G(a) \in A^+$ and $G((1-a^2)^{\frac{1}{2}}) \in A^+$. Hence it follows by Lemma 4·1·2, that

$$G((1-a^2)^{\frac{1}{2}}) = (1-(G(a)^2)^{\frac{1}{2}} .$$

As the same holds for αa whenever $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$, and the binomial expansion for $(1-t)^{\frac{1}{2}} = 1 - \frac{1}{2}t + \sum_{n=2}^{\infty} c_n t^n$ converges absolutely and uniformly on $[0,1]$, it follows that

$$1 - \tfrac{1}{2}\alpha^2 G(a^2) + \sum_{n=2}^{\infty} c_n \alpha^{2n} G(a^{2n}) = 1 - \tfrac{1}{2}\alpha^2 (G(a))^2 + \sum_{n=2}^{\infty} c_n \alpha^{2n} (G(a))^{2n} .$$

Hence

$$G(a^2) - 2\sum_{n=2}^{\infty} c_n \alpha^{2n-2} G(a^{2n}) = (G(a))^2 - 2\sum_{n=2}^{\infty} c_n \alpha^{2n-2} (G(a))^{2n} .$$

Thus, as $\alpha \to 0$, we conclude that $G(a^2) = (G(a))^2$.

Next, by a suitable scalar multiplication, if $b \in A^+ \cap \mathrm{Inv}A$, then $G(b^2) = (G(b))^2$. Finally, given $c \in \mathrm{Her}A$, there exists $\lambda \in \mathbb{R}^+$ such that $c + \lambda \in A^+ \cap \mathrm{Inv}A$. Then

$$\begin{aligned} \lambda^2 + 2\lambda G(c) + G(c^2) &= G((\lambda+c)^2) \\ &= (G(\lambda+c))^2 \\ &= \lambda^2 + 2\lambda G(c) + (G(c))^2 , \end{aligned}$$

and so $G(c^2) = (G(c))^2$.

Our first application is a converse to Theorem 2·4·15. This extends a result of Civin and Yood [24].

COROLLARY 4·1·5. Let A and B be unital JB^*-algebras, and let $F : A \to B$ be a continuous linear map of A onto B such that $F(1) = 1$, $\mathrm{Ker}F$ is an ideal of A , and $F' : B' \to A'$ is an isometry. Then F is a Jordan *-homomorphism.

Proof. As F is continuous, $\mathrm{Ker}F$ is a closed ideal of A , and hence is self-adjoint. Moreover, $A/\mathrm{Ker}F$ is a unital JB^*-algebra, and if $p : A \to A/\mathrm{Ker}F$ is the natural map, then p is a continuous Jordan *-homomorphism, and there exists a well defined one to one continuous linear map of $A/\mathrm{Ker}F$ onto B , such that $F = Gp$. Hence $F' = p'G'$, and so, as F' and p' are isometres, G' is an isometry of B' onto $(A/\mathrm{Ker}F)'$. Thus G is an isometry of $A/\mathrm{Ker}F$ onto B . As $G(1+\mathrm{Ker}F) = 1$, it follows that G is a Jordan *-isomorphism by Theorem 4·1·4. Hence $F = Gp$ is a Jordan *-homomorphism.

We conclude this section with two results concerning auxiliary involutions on JB^*-algebras. This generalises results of Civin and Yood [23], and is included for comparison with the results of section 4, and to give a partial solution of a problem in [12].

COROLLARY 4·1·6. Let A be a unital JB^*-algebra, with defining involution $*$. Let $\dagger$ be any involution on A . Then $\|a^\dagger\| = \|a\|$ for all a in A if and only if $a^{\dagger *} = a^{*\dagger}$ for all a in A .

Proof. Suppose that $a^{\dagger *} = a^{*\dagger}$ for all a in A, and define $G : A \to A$ by $G(a) = a^{\dagger *}$. Then G is a Jordan *-automorphism, so by Theorem 2·3·3, G is an isometry. Hence, by Lemma 2·3·1, we have $\|a^{\dagger}\| = \|a\|$ for all a in A.

Conversely suppose $\|a^{\dagger}\| = \|a\|$ for all a in A. If $G : A \to A$ is defined by $G(a) = a^{\dagger *}$, then G is an isometry of A onto itself, and $G(1) = 1$. Hence, by Theorem 4·1·4, G is a Jordan *-homomorphism, and so

$$a^{\dagger *} = G(a) = (G(a^*))^* = a^{*\dagger **} = a^{*\dagger} ,$$

for all a in A.

Before our final result in this section, we require two Lemmas, which appear in [22].

LEMMA 4·1·7. Let V be a complex linear space, and let $^{\dagger},^{*}$ be two linear involutions on V such that $a^{*\dagger} = a^{\dagger *}$ for all a in V. Then $a^* = a^{\dagger}$ if and only if $x = 0$ is the only solution of $x^{\dagger} = - x^*$.

Proof. Suppose $x^{\dagger} = - x^*$ implies $x = 0$. Let $z \in V$, and let $y = z - z^{\dagger *}$. Then

$$y^* = z^* - z^{\dagger} = -(z^{\dagger} - z^{\dagger *\dagger}) = - y^{\dagger} ,$$

and so $y = 0$. Hence $z = z^{\dagger *}$. The converse is clear.

LEMMA 4·1·8. Let X be a compact Hausdorff space, and let G be a *-automorphism of $C(X)$ such that $G^2 = I$. If G is not the identity, there exists a non-zero self-adjoint element $f \in C(X)$ such that $fG(f) = 0$.

Proof. G induces a homeomorphism t of X onto X such that

$$(Gf)(x) = f(t(x))$$

for all x in X. As $G^2 = I$, it follows that $t^2 = I$. If $G \neq I$, then there exists $x \in X$ such that $t(x) \neq x$. Let U be a neighbourhood of x such that $t(x) \notin \overline{U}$, and so $x \notin \overline{t(U)}$. If $W = U \setminus \overline{t(U)}$, we note that $W \cap t(W) = \emptyset$. As $t(W)$ is an open set containing $t(x)$, by Urysohn's lemma, there exists a continuous real valued function f on X such that $f(t(x)) = 1$, and $f(z) = 0$ for $z \notin t(W)$. Then, for any $z \in X$, we have

$$(fG(f))(z) = f(z)f(t(z)) = 0 .$$

THEOREM 4·1·9. Let A be a unital JB*-algebra with defining involution $*$. If $\dagger$ is an involution on A such that $a^{\dagger *} = a^{*\dagger}$ for all a in A and if $x = 0$ is the only solution of $\{x,x^\dagger,x\} = 0$, then $a^\dagger = a^*$ for all a in A.

<u>Proof</u> By Lemma 4·1·7, it suffices to show that if $x \in A$, and $x^\dagger = - x^*$, then $x = 0$. Let $x = h + ik$ be the natural decomposition of x with respect to the involution $*$. We note that if $s^* = s$, then $(s^\dagger)^* = s^{*\dagger} = s^\dagger$, so that $h + h^\dagger$ and $k + k^\dagger$ are self-adjoint with respect to the involution $*$. As

$$0 = x^\dagger + x^* = (h^\dagger+h) - i(k^\dagger+k) ,$$

it follows that $0 = h^\dagger + h = k^\dagger + k$.

By Corollary 4·1·6, $\dagger$ is continuous, and so, as $h^\dagger = - h$ and $1^\dagger = 1$, we may define a *-automorphism $G : P(1,h) \to P(1,h)$ by $G(a) = a^{*\dagger}$ for a in $P(1,h)$. By Lemmas 4·1·8 and 2·3·1, if G is not the identity, there exists a non-zero self-adjoint element $f \in P(1,h)$ such that $\{f,G(f),f\} = 0$. As G is a *-automorphism, $G(f) = f^\dagger$, and so $\{f,f^\dagger,f\} = 0$. This contradicts the hypothesis, and hence G is the identity. In particular,

$$h^{\dagger} = h^{*} = h ,$$

and so $h = 0$. Similarly $k = 0$, and so $x = 0$, as required.

4.2 Hermitian Operators on JB^{*}-algebras.

In [77], Sinclair used Kadison's theorem on linear isometries in [46] to give an algebraic characterisation of the Hermitian operators on a unital C^{*}-algebra. In this section, we extend this to characterise the Hermitian operators on a unital JB^{*}-algebra, and as a Corollary, we derive a large class of Hermitian operators whose squares are not Hermitian. This will appear in [97]; an alternative approach has been given in [50], [52] and [19].

It follows from Theorem 2·1·4, that if A is a unital JB^{*}-algebra, and $a \in \mathrm{Her}A$, then $L_a \in \mathrm{Her}B(A)$. We now consider a second source of Hermitian elements of $B(A)$.

DEFINITION. Let A be a JB^{*}-algebra. A derivation on A is a linear map $\delta : A \to A$ such that, for all x and y in A ,

$$\delta(x o y) = \delta(x) \, o \, y + x \, o \, \delta(y) \quad .$$

A *-derivation on A is a derivation δ on A such that

$$\delta(x^{*}) = -(\delta(x))^{*}$$

for all x in A .

We note that it has not been assumed that δ is continuous; this will be proved for embedable JB^{*}-algebras in Theorem 4·2·2. First we require some well known results for derivations, see [17] and [43].

LEMMA 4·2·1. Let A be a JB^*-algebra, and let δ be a derivation on A.

(i) If A has a unit, $\delta(1) = 0$.

(ii) For all x, y and z in A,

$$\delta\{x,y,z\} = \{\delta x,y,z\} + \{x,\delta y,z\} + \{x,y,\delta z\} .$$

(iii) δ is the sum of two *-derivations.

(iv) If $\binom{m}{k}$ denotes the binomial coefficient, then for all m in $\mathbb{N}$ and for all x and y in A,

$$\delta^m(x \circ y) = \sum_{k=0}^{m} \binom{m}{k} (\delta^k x)\circ(\delta^{m-k} y).$$

Proof. (i) As $\delta(1) = \delta(1^2) = 2\delta(1)$, it follows that $\delta(1) = 0$.

(ii) As $\{x,y,z\} = (x\circ y)\circ z - (z\circ x)\circ y + (y\circ z)\circ x$, it follows that

$$\begin{aligned}\delta\{x,y,z\} &= (x\circ y)\circ\delta z + (\delta x\circ y)\circ z + (x\circ\delta y)\circ z \\ &\quad - (z\circ x)\circ\delta y - (\delta z\circ x)\circ y - (z\circ\delta x)\circ y \\ &\quad + (y\circ z)\circ\delta x + (\delta y\circ z)\circ x + (y\circ\delta z)\circ x \\ &= \{\delta x,y,z\} + \{x,\delta y,z\} + \{x,y,\delta z\} .\end{aligned}$$

(iii) Let $\delta^{(1)}(x) = \delta(x) - (\delta(x^*))^*$, and $\delta^{(2)}(x) = i(\delta(x)+(\delta(x^*))^*)$ for x in A. Then $\delta^{(1)}$ and $\delta^{(2)}$ are *-derivations on A such that

$$\delta = \tfrac{1}{2}(\delta^{(1)} - i\delta^{(2)}) .$$

(iv) This follows by induction.

THEOREM 4·2·2. Let A be an embedable JB^*-algebra, and let δ be a derivation on A. Then δ is continuous.

Proof. First, we may assume that A has a unit, for, it not, we may define $\delta^{(1)}: A\oplus\mathbb{C} \to A\oplus\mathbb{C}$ by $\delta^{(1)}(x+\mu) = \delta(x)$, and it is easy to check that $\delta^{(1)}$ is a derivation. By Lemma 4·2·1 (iii), it suffices to assume that $i\delta$ is a *-derivation. Moreover, as it is enough to show that δ is continuous on HerA, then, by the closed graph theorem, it suffices to show that, given a sequence $\{x_n\}$ in $\mathrm{Her}A\setminus\{0\}$

such that $x_n \to 0$ and $\delta(x_n) \to a + ib$, with a and b in $\mathrm{Her}A$, then $a = b = 0$.

As $i\delta$ is a *-derivation, $\delta(x^*) = -i(i\delta(x^*)) = i(i\delta(x))^* = (\delta(x))^*$ for all x in A, and so

$$a - ib = (a+ib)^* = \lim(\delta(x_n))^* = \lim\delta(x_n^*) = \lim\delta(x_n) = a + ib.$$

Thus $b = 0$.

Suppose that $a \neq 0$. By considering $\{kx_n\}$ for a suitable $k \in \mathbb{R}$ if necessary, we may assume $1 \in \sigma(a)$. Define $f : \mathbb{R} \to \mathbb{R}^+$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2} \\ 2(x-\frac{1}{2}) & \text{if } \frac{1}{2} \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}.$$

Then f is continuous, and, if we identify $P(1,a)$ with $C(\sigma(a))$, $h = f|_{\sigma(a)}$ is a positive element of norm one in $P(1,a)$ such that

$$\{h,a,h\} \geq \tfrac{1}{2}h^2 .$$

Let $y_n = x_n + 3\|x_n\|1$. Then $y_n \to 0$, $\delta y_n = \delta x_n$, and so

$$\delta\{h,y_n,h\} = 2\{\delta h,y_n,h\} + \{h,\delta y_n,h\}$$
$$\to \{h,a,h\} .$$

Hence, there exists $m \in \mathbb{N}$ such that, for $n \geq m$,

$$\|\delta\{h,y_n,h\} - \{h,a,h\}\| < 1/8 . \qquad \text{(i)}$$

Now, as $y_n \leq \|y_n\| \leq 4\|x_n\|$, and U_h is a positive operator,

$$U_h y_n \leq 4\|x_n\|h^2$$

and so

$$(8\|x_n\|)^{-1}\{h,y_n,h\} \leq \{h,a,h\} . \qquad \text{(ii)}$$

Moreover, $\|x_n\| + x_n \geq 0$, so that

$$(4\|x_n\|)^{-1}\{h,y_n,h\} \geq (4\|x_n\|)^{-1}\, 2\|x_n\|h^2 = \tfrac{1}{2}h^2$$

and hence

$$\|(4\|x_n\|)^{-1}\{h,y_n,h\}\| \geq \tfrac{1}{2}\|h^2\| = \tfrac{1}{2} . \qquad \text{(iii)}$$

Let $d = \{h,y_m,h\}$. By (iii) , there is a multiplicative linear functional ϕ on $P(1,d)$ such that $\phi(d) \geq 2\|x_m\|$. As $\phi \in D(P(1,d),1)$, there exists $\psi \in D(A,1)$ such that $\psi|_{P(1,d)} = \phi$. We let $K = \{x \in A : \psi(xox^*) = 0\}$ and note that, as ϕ is multiplicative on $P(1,d)$, $K \cap P(1,d) = \mathrm{Ker}\phi$. Hence, there exists $u, v \in (P(1,d))^+$ such that $u, v \in K \cap P(1,d)$ and $d - \phi(d)1 = u^2 - v^2$.

Now, for all $\lambda \in \mathbb{R}$, and all $w \in A$, we have

$$\begin{aligned}&\psi(uou^*) + \lambda.\psi(uow^*+wou^*) + \lambda^2.\psi(wow^*)\\&= \psi((u+\lambda w)o(u+\lambda w)^*)\\&\geq 0 .\end{aligned}$$

Hence,

$$(\psi(uow^*+wou^*))^2 \leq 4\psi(uou^*)\psi(wow^*) .$$

As $u = u^* \in K$, for all w in A , we have $\psi(uo(w+w^*)) = 0$. If we replace w by iw in the above equality, it follows that $i\,\psi(uo(w-w^*)) = 0$, and so $\psi(uow) = 0$ for all w in A . A similar argument shows that $\psi(vow) = 0$ for all w in A , and, in particular

$$\psi(uo\delta(u)) = \psi(vo\delta(v)) = 0 .$$

Hence

$$\begin{aligned}\psi(\delta(d)) &= \psi(\delta(d-\phi(d)1))\\&= \psi(\delta(u^2-v^2))\\&= 0 . \qquad\qquad (iv)\end{aligned}$$

By (i) and (iv) ,

$$|\psi(\{h,a,h\})| < 1/8 .$$

However, this contradicts (ii), as

$$\begin{aligned}|\psi\{h,a,h\}| &\geq \tfrac{1}{2}\phi((4\|x_m\|)^{-1} d)\\&\geq \tfrac{1}{2}\cdot\tfrac{1}{2} .\end{aligned}$$

Thus $a = 0$, and this completes the proof of the theorem.

Before we derive the analogue of Sinclair's results for Hermitian operators on B^*-algebras, we consider the connections between *-derivations and *-automorphisms. This is well known for semi-simple finite dimensional Jordan algebras, and the proof given in [17] is also valid for JB^*-algebras.

THEOREM 4·2·3. Let A be a unital JB^*-algebra.

(i) If δ is a *-derivation, then $\exp it\delta$ is a *-automorphism for all $t \in \mathbb{R}$.

(ii) Conversely, if $\psi \in B(A)$ and $\exp it\psi$ is a *-automorphism for all $t \in \mathbb{R}$, then ψ is a *-derivation.

Proof. (i) $\exp it\delta$ is a one to one continuous linear map of A onto A. If x and y are in A, then

$$\begin{aligned}(\exp it\delta)(xoy) &= \sum_{n=0}^{\infty}(n!)^{-1}(it\delta)^n(xoy)\\ &= \sum_{n=0}^{\infty}(n!)^{-1}(it)^n \sum_{k=0}^{n}\binom{n}{k}(\delta^k x)o(\delta^{n-k}y)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}((k!)^{-1}(it\delta)^k x)o(((n-k)!)^{-1}(it\delta)^{n-k}y)\\ &= (\sum_{r=0}^{\infty}(r!)^{-1}(it\delta)^r x)o(\sum_{s=0}^{\infty}(s!)^{-1}(it\delta)^s y)\\ &= ((\exp it\delta)x)o((\exp it\delta)y)\ .\end{aligned}$$

Moreover,

$$\begin{aligned}((\exp it\delta)(x^*))^* &= (\sum_{n=0}^{\infty}(n!)^{-1}((it\delta)^n x^*)^*\\ &= (\sum_{n=0}^{\infty}(n!)^{-1}(it)^n(-1)^n(\delta^n x)^*)^*\\ &= \sum_{n=0}^{\infty}(n!)^{-1}(it\delta)^n(x)\\ &= (\exp it\delta)(x)\ .\end{aligned}$$

Hence $\exp it\delta$ is a *-automorphism of A.

(ii) As $\exp it\psi$ is a *-automorphism, given x and y in A, we have

$$\begin{aligned}&x\ o\ y + it\psi(xoy) + \sum_{n=2}^{\infty}(it\psi)^n(xoy)\\ &= (x + it\psi(x) + \sum_{n=2}^{\infty}(it\psi)^n x)o(y+it\psi(y) + \sum_{n=2}^{\infty}(it\psi)^n y)\ .\end{aligned}$$

If we equate the power of t of degree 1 in the equality, we obtain

$$\psi(xoy) = \psi(x)o\, y + x\, o\psi(y) .$$

Similarly, if we equate the power of t of degree 1 in the equation

$$x^* + it\psi(x^*) + \sum_{n=2}^{\infty}(it\psi)^n(x^*) = x^* - it(\psi(x))^* + (\sum_{n=2}^{\infty}(it\psi)^n(x))^* ,$$

we conclude that

$$\psi(x^*) = -(\psi(x))^* .$$

THEOREM 4·2·4. Let A be a unital JB^*-algebra. If $a \in \mathrm{Her}A$ and δ is a *-derivation on A, then $L_a \in \mathrm{Her}B(A)$ and $\delta \in \mathrm{Her}B(A)$. Conversely, if $\lambda \in \mathrm{Her}B(A)$, then there is a unique $b \in \mathrm{Her}A$ and a unique *-derivation ψ on A such that $\lambda = L_b + \psi$.

Proof. If $a \in \mathrm{Her}A$, then $L_a \in \mathrm{Her}B(A)$ by Theorem 2·1·4. If δ is a *-derivation on A, then $\delta \in \mathrm{Her}B(A)$ by Theorems 4·2·3 and 2·3·3, and Corollary 2·2·2.

Conversely, suppose $\lambda \in B(A)$. Let $b = \lambda(1)$. Given $f \in D(A,1)$, we define $F \in (B(A))'$ by $F(\sigma) = f(\sigma(1))$ for $\sigma \in B(A)$. As $F(I) = 1$ and

$$|F(\sigma)| \leq \|\sigma(1)\| \leq \|\sigma\| ,$$

it follows that $F \in D(B(A),I)$. Hence $f(b) = f(\lambda(1)) = F(\lambda) \in \mathbb{R}$, and so $b \in \mathrm{Her}A$. Thus $\psi = \lambda - L_b \in \mathrm{Her}B(A)$.

As $\psi(1) = 0$, it follows that $\exp it\psi$ is a linear isometry of A onto A which takes 1 onto 1. Hence, by Theorems 4·1·4 and 4·2·3, ψ is a *-derivation.

Finally, as every derivation of A maps 1 to 0, it follows that the decomposition of λ is unique.

An alternative characterisation of the Hermitian operators on a unital JB^*-algebra is given in the following Theorem. Kaup [50] also obtained this result, as we point out in Theorem 4·4·3.

THEOREM 4·2·5. Let A be a unital JB^*-algebra, and let $\lambda \in B(A)$. Then $\lambda \in \mathrm{HerB}(A)$ if and only if

$$\lambda\{b,c^*,d\} = \{\lambda b,c^*,d\} - \{b,(\lambda c)^*,d\} + \{b,c^*,\lambda d\} \qquad (\dagger)$$

for all b, c and d in A.

Proof. If $\lambda \in \mathrm{HerB}(A)$, then $\lambda = L_a + \delta$ where $a \in \mathrm{Her}A$ and δ is a *-derivation, by Theorem 4·2·4, and so it suffices to show that (†) holds for L_a and δ. However, δ satisfies (†) by Lemma 4·2·1, while from [43] p.37, for all b, c and d in A, we have

$$\begin{aligned} L_a\{b,c^*,d\} &= \{L_a b,c^*,d\} - \{b,L_a(c^*),d\} + \{b,c^*,L_a d\} \\ &= \{L_a b,c^*,d\} - \{b,(L_a c)^*,d\} + \{b,c^*,L_a d\} \end{aligned}$$

as $a \in \mathrm{Her}A$.

Conversely, suppose $\lambda \in B(A)$ and λ satisfies (†). Let $a = \lambda(1)$. Choosing $b = c = d = 1$, we have $\lambda(1) = \lambda(1) - (\lambda(1))^* + \lambda(1)$, so that a is self-adjoint. Hence, by Theorem 2·1·4, $L_a \in \mathrm{HerB}(A)$. If $\psi = \lambda - L_a$, then $\psi(1) = 0$, and by the previous part of the proof ψ also satisfies (†). Taking $d = 1$, we deduce that for all b and c in A,

$$\psi(b \circ c^*) = \psi(b) \circ c^* - b \circ (\psi(c))^* .$$

In particular $\psi(c^*) = -(\psi(c))^*$, and so, for all b and c in A, we have

$$\psi(b \circ c) = \psi(b) \circ c + b \circ \psi(c) .$$

Thus ψ is a *-derivation, and so $\lambda = L_a + \psi \in \mathrm{HerB}(A)$ by Theorem 4·2·4.

COROLLARY 4·2·6. Let A be a unital JB^*-algebra, and let $\lambda \in B(A)$. Then the following are equivalent :

(i) $\lambda \in \mathrm{HerB}(A)$,

(ii) $\lambda\{a,b^*,a\} = 2\{\lambda a,b^*,a\} - \{a,(\lambda b)^*,a\}$ for all a and b in A ,

(iii) $\lambda\{a,a^*,a\} = 2\{\lambda a,a^*,a\} - \{a,(\lambda a)^*,a\}$ for all a in A .

We shall not prove the above Corollary, which is obtained by linearisation of (†) , as it is included only for comparison with Theorem 4·4·3. Instead, we derive from Theorem 4·2·5 a characterisation of those $a \in \mathrm{Her}A$ such that $(L_a)^2 \in \mathrm{HerB}(A)$.

THEOREM 4·2·7. Let A be a unital JB^*-algebra, and let $a \in \mathrm{Her}A$. Then $(L_a)^2 \in \mathrm{HerB}(A)$ if and only if $a \in Z(A)$.

Proof. As $(L_a)^2 = \frac{1}{2}(U_a + L_{a^2})$, it follows that $(L_a)^2 \in \mathrm{HerB}(A)$ if and only if $U_a \in \mathrm{HerB}(A)$. Suppose first that $U_a \in \mathrm{HerB}(A)$. Let $h \in \mathrm{Her}A$, and let $x, y \in (P(1,h))^+$ such that $h = x^2 - y^2$. Then

$$U_a\{x,x,1\} = \{U_a x,x,1\} - \{x,(U_a x)^*,1\} + \{x,x,U_a 1\} .$$

Since a and x are self-adjoint, $U_a x = (U_a x)^*$, and so

$$\{a,x^2,a\} = x^2 \circ a^2 .$$

Similarly, $\{a,y^2,a\} = y^2 \circ a^2$, so that

$$\{a,h,a\} = h \circ a^2 .$$

Hence, by Theorem 3·4·1, $a \in Z(A)$.

Conversely, if $a \in Z(A)$, and $b \in \mathrm{Her}A$, then

$$U_a(b) = L_{a^2}(b) ,$$

and so $(L_a)^2 \in \mathrm{HerB}(A)$.

4.3 The general Kaplansky conjecture.

In this section, we consider a more general problem than that of section 4·1. Here, we characterise all isometries between embedable

JB^*-algebras. It is a modification of work of Harris [40], and an alternative approach is due to Kaup [50].

Notation If A is a unital JB^*-algebra, we let

$$K(A) = \{x+iy : x,y \in \mathrm{Her}A \text{ and } y \in A^+ \cap \mathrm{Inv}\, A\},$$

and we let A_0 denote the open unit ball of A.

$K(A)$ can be regarded as a generalised "upper half-plane". Our first aim in this section is to show that A_0 is a bounded symmetric homogeneous domain.

THEOREM 4·3·1. Let A be a unital JB^*-algebra. Define $s : A_0 \to A$ and $t : K(A) \to A$ by $s(a) = i(a+1)(1-a)^{-1}$ and $t(b) = (b-i)(b+i)^{-1}$. Then s is a biholomorphic map of A_0 onto $K(A)$ with inverse t, $Ds(a) = 2iU_{(1-a)^{-1}}$ and $Dt(b) = 2iU_{(b+i)^{-1}}$.

Proof If $a \in A_0$, then $1 - a$ is invertible, so $s(a)$ is well defined. Also, if $b \in K(A)$, $0 \notin V(b+i)$, and so $t(b)$ is well defined.

Let $a \in A_0$. As $s(a)$ and $(s(a))^*$ are in $P(1,a,a^*)$, in order to show that $s(a) \in K(A)$, it suffices to show that if $\tilde{s} = s|_{P(1,a,a^*)}$, then $\tilde{s}(a) \in K(P(1,a,a^*))$. By Theorem 2·3·5, we may regard $P(1,a,a^*)$ as a JC^*-algebra, and so

$$\begin{aligned}\tfrac{1}{2}i^{-1}(\tilde{s}(a)-(\tilde{s}(a))^*) &= \tfrac{1}{2}((a+1)(1-a)^{-1} + (1-a^*)^{-1}(a^*+1)) \\ &= \tfrac{1}{2}(1-a^*)^{-1}((1-a^*)(a+1)+(a^*+1)(1-a))(1-a)^{-1} \\ &= (1-a^*)^{-1}(1-a^*a)(1-a)^{-1} .\end{aligned}$$

As $\|a\| < 1$, $1 - a^*a \in A^+ \cap \mathrm{Inv}\, A$, and hence $\tfrac{1}{2}i^{-1}(\tilde{s}(a)-(\tilde{s}(a))^*) \in A^+ \cap \mathrm{Inv}\, A$. Thus $\tilde{s}(a) \in K(P(1,a,a^*))$.

Let $b \in K(A)$. It similarly suffices to show that if

$\tilde{t} = t|_{P(1,b,b^*)}$ then $\|\tilde{t}(b)\| < 1$, and again we regard $P(1,b,b^*)$ as a JC*-algebra. So

$$1 - \tilde{t}(b)(\tilde{t}(b))^* = 1 - (1-2i(b+i)^{-1})(1+2i(b^*-i)^{-1})$$
$$= (b+i)^{-1}((b+i)(b^*-i) - (b-i)(b^*+i))(b^*-i)^{-1}$$
$$= 2(b+i)^{-1}(i(b^*-b))(b^*-i)^{-1} .$$

As $i^{-1}(b-b^*) \in A^+ \cap \mathrm{Inv}A$, it follows that $1 - \tilde{t}(b)(\tilde{t}(b))^* \in A^+ \cap \mathrm{Inv}A$, and so $\|\tilde{t}(b)\| < 1$. Hence, t maps $K(A)$ into A_0.

It is easy to verify that $\tilde{s}\tilde{t}(b) = b$ and $\tilde{t}\tilde{s}(a) = a$, so that s and t are inverses. As $s(a) = 2i(1-a)^{-1} - i$ and $t(b) = 1 - 2i(b+i)^{-1}$, it follows from Lemma 1·2·2 that s and t are holomorphic, and $Ds(a) = 2iU_{(1-a)^{-1}}$ and $Dt(b) = 2iU_{(b+i)^{-1}}$.

LEMMA 4·3·2. Let A be a unital JB*-algebra, and let $b \in K(A)$, with standard decomposition $x + iy$. Define $p_b : K(A) \to A$ by

$$p_b(c) = x + U_{y^{\frac{1}{2}}}(c) .$$

for c in $K(A)$. Then p_b is a biholomorphic map of $K(A)$ onto itself, $Dp_b(c) = U_{y^{\frac{1}{2}}}$, and $p_b(i) = b$.

<u>Proof</u>. Let $c \in K(A)$, and let $h + ik$ be the standard decomposition of c. Then $p_b(c) = x + U_{y^{\frac{1}{2}}}(h) + iU_{y^{\frac{1}{2}}}(k)$, so that $p_b(c) \in K(A)$. Thus p_b maps $K(A)$ into $K(A)$. Moreover, if we define $q_b(c) = U_{y^{-\frac{1}{2}}}(c-x)$, $p_b q_b = q_b p_b = I$, and q_b maps $K(A)$ into $K(A)$. Hence, p_b is a one to one map of $K(A)$ onto $K(A)$. It is clear that p_b is biholomorphic with $Dp_b(c) = U_{y^{\frac{1}{2}}}$ and that $p_b(i) = b$.

COROLLARY 4·3·3. Let A be a unital JB*-algebra, and let $a \in A_0$.

(i) The map $R_a : A_0 \to A_0$ defined by $R_a = tp_{s(a)}s$ is a biholomorphic map of A_0 onto A_0 such that $R_a(0) = a$.

(ii) If $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$, and $a = \alpha 1$, then $R_a(z) = (\alpha+z)(1+\alpha z)^{-1}$ and $DR_a(z) = (1-\alpha^2)U_{(1+\alpha z)^{-1}}$ for all z in A_o.

Proof. (i) It follows by Theorem 4·3·1 and Lemma 4·3·2 that R_a is a biholomorphic map of A_o onto itself. Further,

$$R_a(0) = tp_{s(a)}s(0) = tp_{s(a)}(i) = ts(a) = a .$$

(ii) As $s(a) = i(1+\alpha)(1-\alpha)^{-1}$, $p_{s(a)}(c) = (1+\alpha)(1-\alpha)^{-1}(c)$ for c in $K(A)$. Hence, for $z \in A_o$, we have

$$\begin{aligned} tp_{s(a)}s(z) &= ((1+\alpha)(1-\alpha)^{-1}s(z)-i)((1+\alpha)(1-\alpha)^{-1}s(z)+i)^{-1} \\ &= ((1+\alpha)(1+z)-(1-\alpha)(1-z))((1+\alpha)(1+z)+(1-\alpha)(1-z))^{-1} \\ &= (\alpha+z)(1+\alpha z)^{-1} . \end{aligned}$$

Thus $R_a(z) = (\alpha+z)(1+\alpha z)^{-1} = \alpha^{-1}(1+\alpha z + \alpha^2-1)(1+\alpha z)^{-1}$, and so

$$\begin{aligned} DR_a(z) &= -\alpha^{-1}(\alpha^2-1)\, \alpha\, U_{(1+\alpha z)^{-1}} \\ &= (1-\alpha^2)U_{(1+\alpha z)^{-1}} . \end{aligned}$$

The maps R_a defined in Corollary 4·3·3 are the "obvious" analogues of Möbius transformations, at least when a is central. Better candidates for the special case may be found in [41].

Before we apply Corollary 4·3·3 to derive our main results in this section, we require a Lemma.

LEMMA 4·3·4. Let X be a complex Banach space with open unit ball W, and suppose that $f : W \to W$ is biholomorphic. Suppose also that Y is a closed subspace of X such that if $g = f|_{Y\cap W}$, then g maps $Y \cap W$ onto $Y \cap W$. Then g is a biholomorphic map of $Y \cap W$ onto $Y \cap W$ such that

$$Dg(a)(c) = Df(a)(c)$$

for all $a \in Y \cap W$ and all $c \in Y$.

Proof. Let $a \in Y \cap W$, and let $d \in Y$ be such that $\|d\| = 1$. As f is holomorphic, for all $\eta \in \mathbb{R}^+ \setminus \{0\}$, there exists $\delta \in \mathbb{R}^+ \setminus \{0\}$ such that whenever $\|h\| \leq \delta$, we have

$$\|f(a+h) - f(a) - Df(a)(h)\| < \eta\|h\| .$$

In particular, for $h = \delta d$,

$$\|\delta^{-1}(f(a+h) - f(a)) - Df(a)(d)\| < \eta .$$

Hence, as Y is a closed subspace, $Df(a)(d) \in Y$, and so, by a suitable scalar multiplication, we conclude that $Df(a)|_Y \in B(Y)$. Moreover, for all $h \in Y$ with $\|h\| \leq \delta$, we have

$$\|g(a+h) - g(a) - Df(a)(h)\| < \eta\|h\| ,$$

and so g is holomorphic, and $Dg(a)(c) = Df(a)(c)$ for all $a \in Y \cap W$, and all $c \in Y$.

Finally, f^{-1} also satisfies the hypothesis of the Lemma, and $f^{-1}|_{Y \cap W} = (f|_{Y \cap W})^{-1} = g^{-1}$. It follows by the above that g^{-1} is also holomorphic. Hence g is biholomorphic, as required.

THEOREM 4·3·5. If A is a unital JB*-algebra, and $F:A \to A$ is a linear isometry of A onto A, then $F(1)$ is unitary.

Proof. Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < 1$, and let $a = \alpha 1$. For $b \in A_0$, we let $R_b : A_0 \to A_0$ be the biholomorphic map of A_0 onto itself, with $R_b(0) = b$, defined in Corollary 4·3·3. If $h = (R_{F(a)})^{-1}FR_a$, then h is a biholomorphic map of A_0 onto A_0 by the chain rule, and $h(0) = 0$. Hence, by Corollary 1·4·4, h is the restriction of a linear isometry of A. In particular, for all x and y in A_0, it follows that $Dh(x) = Dh(y)$.

We now apply the chain rule to compute $Dh(x)$ and $Dh(y)$, where $x = -\alpha 1$, and $y = 0$.

$$Dh(x) = [D(R_{F(a)})^{-1}(FR_a(x))]F[DR_a(x)]$$

$$= [D(R_{F(a)})^{-1}(0)]F(1-\alpha^2)U_{(1-\alpha^2)^{-1}}$$

$$= (1-\alpha^2)^{-1}[D(R_{F(a)})^{-1}(0)]F .$$

$$Dh(y) = [D(R_{F(a)})^{-1}F(a)]F[DR_a(0)]$$

$$= (1-\alpha^2)[D(R_{F(a)})^{-1}F(a)]F .$$

As F is a linear isometry of A onto itself, it follows that

$$(1-\alpha^2)^{-1}D(R_{F(a)})^{-1}(0) = (1-\alpha^2)D(R_{F(a)})^{-1}F(a) .$$

Let $Y = P(1,F(a),(F(a))^*)$. As $(R_{F(a)})^{-1}$ maps $Y \cap A_0$ onto $Y \cap A_0$, it follows by Lemma 4·3·4 that if $g = (R_{F(a)})^{-1}|_{Y\cap A_0}$, then g is biholomorphic and

$$Dg(b)(c) = D(R_{F(a)})^{-1}(b)(c)$$

for all $b \in Y \cap A_0$ and all c in Y .

By Theorem 2·3·5, we may regard Y as a JC*-algebra, so that, by Corollary 1·4·7, $g = kT_{-g^{-1}(0)}$, where k is a linear isometry of Y onto Y , and

$$T_p(q) = (1-pp^*)^{-\frac{1}{2}}(p+q)(1+p^*q)^{-1}(1-p^*p)^{\frac{1}{2}}$$

for p and q in $Y \cap A_0$. As $g^{-1}(0) = ((R_{F(a)})^{-1}|_{Y\cap A_0})^{-1}(0)=R_{F(a)}(0)=F(a)$, it follows by the chain rule that

$$D(R_{F(a)})^{-1}(b)(c) = k[DT_{-F(a)}(b)](c)$$
$$= k(1-F(a)(F(a))^*)^{\frac{1}{2}}(1-b(F(a))^*)^{-1}c(1-(F(a))^*b)^{-1}(1-(F(a))^*F(a))^{\frac{1}{2}}$$

for all b in $Y \cap A_0$ and all c in Y . Thus

$$D(R_{F(a)})^{-1}(0)(c) = k(1-F(a)(F(a))^*)^{\frac{1}{2}}c(1-(F(a))^*F(a))^{\frac{1}{2}}$$

and

$$D(R_{F(a)})^{-1}(F(a))(c) = k(1-F(a)(F(a))^*)^{-\frac{1}{2}}c(1-(F(a))^*F(a))^{-\frac{1}{2}}$$

for all c in Y . Hence, as k is one to one, it follows that

$$(1-\alpha^2)^{-1}(1-F(a)(F(a))^*)^{\frac{1}{2}}(1-(F(a))^*F(a))^{\frac{1}{2}} =$$
$$= (1-\alpha^2)(1-F(a)(F(a))^*)^{-\frac{1}{2}}(1-(F(a))^*F(a))^{-\frac{1}{2}} ,$$

so that

$$(1-F(a)(F(a))^*)(1-(F(a))^*F(a)) = (1-\alpha^2)^2 .$$

Hence,

$$1 - \alpha^2(F(1)(F(1))^*+(F(1))^*F(1)) + \alpha^4 F(1)((F(1))^*)^2F(1) = 1 - 2\alpha^2 + \alpha^4 ,$$

and so

$$F(1)\circ(F(1))^* - 1 = \tfrac{1}{2}\alpha^2(F(1)((F(1))^*)^2F(1)-1) .$$

Thus, as $\alpha \to 0$, we conclude that $F(1)\circ(F(1))^* = 1$. Hence, by Lemma 4·1·1, $F(1)$ is unitary.

A simpler proof of Theorem 4·3·5 is available when $F(1)$ is self-adjoint, since, as $F(1)$ is an extreme point of A_1, it follows that $(F(1))^2 = 1$ by Corollary 3·3·10.

THEOREM 4·3·6. Let A be an embedable JB^*-algebra, and let $F : A \to A$ be a linear isometry of A onto itself. Then, for all x in A,

$$F\{x,x^*,x\} = \{F(x),(F(x))^*,F(x)\} .$$

Proof. By passing to A'' if necessary, we may assume A has a unit. By Theorem 4·3·5, $F(1)$ is unitary, and so $P(1,F(1),(F(1))^*)$ is a commutative B^*-algebra. Hence, by the extended functional calculus, there exists $h \in Her(P(1,F(1),(F(1))^*))''$ such that $F(1) = \exp ih$. We may regard $(P(1,F(1),(F(1))^*))''$ as a subalgebra of A'', so if we define $G : A'' \to A''$ by

$$G(x) = \{\exp -\tfrac{1}{2}ih, F''(x), \exp -\tfrac{1}{2}ih\}$$

for x in A'', it follows that G is a linear isometry of A'' onto itself, and $G(1) = 1$. By Theorems 3·3·4 and 4·1·4, G is a Jordan *-automorphism, and so $G\{x,x^*,x\} = \{G(x),(G(x))^*,G(x)\}$ for all x in A''. Hence

$$\{\exp -\tfrac{1}{2}ih, F''\{x,x^*,x\}, \exp -\tfrac{1}{2}ih\}$$
$$=\{\{\exp -\tfrac{1}{2}ih,F''(x),\exp -\tfrac{1}{2}ih\},\{\exp \tfrac{1}{2}ih,(F''(x))^*,\exp \tfrac{1}{2}ih\},\{\exp -\tfrac{1}{2}ih,F''(x),\exp -\tfrac{1}{2}ih\}\}$$

$= \{\exp -\tfrac{1}{2}ih, \{F''(x), (F''(x))^*, F''(x)\}, \exp -\tfrac{1}{2}ih\}$,

for all x in A''. In particular, $F\{x,x^*,x\} = \{F(x),(F(x))^*,F(x)\}$ for all x in A.

We remark that Kaup and his colleagues have obtained similar results from their constructions associated with bounded symmetric homogeneous domains [18], [19], [50], [51], [52], [53]. We shall indicate some of their results in the following section.

4.4. Recent results on bounded symmetric homogeneous domains.

In the previous section, we showed that the open unit ball of a unital JB*-algebra is a bounded symmetric homogeneous domain. Here, we summarise, without giving the proofs, some of the work recently done on these domains.

The finite dimensional bounded symmetric homogeneous domains were classified using Lie algebra and Lie group theory by Cartan in [20]. Later, Koecher [55] and Loos [56] obtained the same results using Jordan algebra theory. In the following Theorem we give their results. For r in $\mathbb{N}$, we assume $\mathbb{C}^r$ has the usual inner product, and if A is a matrix, we let A^t denote the transpose of A. Moreover, we let $\mathbb{K}\oplus i\mathbb{K}$ denote the complexification of $\mathbb{K}$, the Cayley algebra over $\mathbb{R}$, and let $\bar{\alpha}$ denote the conjugate of α in $\mathbb{K}\oplus i\mathbb{K}$.

THEOREM 4·4·1. Every bounded symmetric homogeneous domain $D \subseteq \mathbb{C}^n$ is biholomorphically equivalent to a product of irreducible bounded symmetric homogeneous domains. Every irreducible bounded symmetric homogeneous domain is equivalent to one of the following list for some r and s in $\mathbb{N}$.

Notation	Domain	Dimension
$I_{r,s}$.	$\{a \in B(\mathbb{C}^r,\mathbb{C}^s):\|a\| < 1\}$	rs
II_r	$\{a \in B(\mathbb{C}^r):a^t = -a,\|a\| < 1\}$	$\frac{1}{2}r(r-1)$
III_r	$\{a \in B(\mathbb{C}^r):a^t = a,\|a\| < 1\}$	$\frac{1}{2}r(r+1)$
IV_r	$\{a \in \mathbb{C}^r:a^*a < \frac{1}{2}(1+\lvert a^ta\rvert^2) < 1\}$	r
V	$\{a \in M_3^8\oplus iM_3^8:\|a\| < 1, a = \begin{bmatrix} 0 & \alpha & \beta \\ \bar{\alpha} & 0 & 0 \\ \bar{\beta} & 0 & 0 \end{bmatrix}$ where $\alpha,\beta \in \mathbb{K}\oplus i\mathbb{K}\}$	16
VI	$\{a \in M_3^8\oplus iM_3^8:\|a\| < 1\}$	27

The norm on $M_3^8\oplus iM_3^8$ is that given by Theorem 2·4·7. By [40] §2, the domain IV_r is the open unit ball of a C^*-triple system, and so every finite dimensional bounded symmetric homogeneous domain is biholomorphically equivalent to the open unit ball of a subspace of a finite dimensional JB^*-algebra. This subspace need not be a subalgebra, but in each case, the subspace is closed under the product $\{a,b^*,c\}$. This leads us to the following definition due to Loos [56] and Kaup [50].

DEFINITION. Let E be a complex Banach space. A Hermitian Jordan triple system on E is a map $\{\ ,\ ^*,\ \} : E\times E\times E \to E$ such that, for all a, b, c and d in E,

(i) $a \to \{a,b^*,c\}$ and $c \to \{a,b^*,c\}$ are continuous linear maps and $b \to \{a,b^*,c\}$ is a continuous conjugate linear map ;

(ii) $\{a,b^*,c\} = \{c,b^*,a\}$;

(iii) $a \,\square\, \{b,c^*,d\}^* - \{a,b^*,c\} \,\square\, d^* = (a \,\square\, d^*)(c \,\square\, b^*)-(c \,\square\, b^*)(a \,\square\, d^*)$;

where $x \,\square\, y^* \in B(E)$ is defined by $(x \,\square\, y^*)(z) = \{x,y^*,z\}$ for x, y and z in E ;

(iv) $a \,\square\, a^* \in HerB(E)$.

Conditions (i), (ii) and (iii) are purely algebraic, and are satisfied by any subspace K of a complex Jordan algebra with involution J, such that $\{a,b^*,c\} \in K$ for all a, b and c in K where $\{\ ,\ ^*,\ \}$ is the normal triple product on J. In [50], Kaup showed that Hermitian Jordan triple systems are the natural structures associated with complex Banach manifolds. We state the main result of [50] in the next Theorem without defining all the concepts involved.

THEOREM 4·4·2. The category of simply connected, symmetric complex Banach manifolds with base point is equivalent to the category of Hermitian Jordan triple systems.

A full analysis of Hermitian Jordan triple systems seems impossible at present. However, Kaup does single out a subclass for which some theory is possible. (In order to be consistant with our previous notation, we do not use the same name as he does for this subclass.)

DEFINITION. A Hermitian Jordan triple system on E is called a B^*-triple system if

(i) $\sigma(a \,\square\, a^*) \geq 0$ for all a in E.

(ii) $\|a \,\square\, a^*\| = \|a\|^2$ for all a in E.

As it follows that $\|\{a,a^*,a\}\| = \|a\|^3$ for all a in a B^*-triple system, whenever A is a Banach Jordan algebra with involution such that A with the natural Jordan triple product is a B^*-triple system, then A is a JB^*-algebra. On the other hand, it is shown in [19] that every unital JB^*-algebra is a B^*-triple system. We now summarise further properties of B^*-triple systems from [19] and [52].

THEOREM 4·4·3. Let E be a B^*-triple system.

(i) The open unit ball of E is a bounded symmetric homogeneous domain.

(ii) If F is a continuous one to one map of E onto E, then F is a linear isometry if and only if $F\{a,a^*,a\} = \{F(a),(F(a))^*,F(a)\}$ for all a in E.

(iii) If $G \in B(E)$, then $G \in HerB(E)$ if and only if

$$G\{a,a^*,a\} = 2\{G(a),a^*,a\} - \{a,(G(a))^*,a\}$$

for all a in E.

(iv) Let $a \in E$ such that $\{a,a^*,a\} = a$. (Such an element is called a tripotent). Then if we define $\wedge : E\times E \to E$ by

$$b \wedge c = \{b,a^*,c\}$$

for b and c in E, $(E,\wedge)$ is a Jordan algebra, and a is an idempotent in E.

(v) a is an extreme point of E_1 if and only if a is a tripotent such that $0 \notin \sigma(a \,\square\, a^*)$.

By the remarks after Theorem 4·4·1, and Theorem 4·4·3, every finite dimensional B^*-triple system may be embedded as a B^*-triple subsystem of a finite dimensional JB^*-algebra. We do not know if this is true for infinite dimensional B^*-triple systems, but nevertheless, with the results we have available, JB^*-algebras are probably the easiest B^*-triple systems which cannot always be embedded as C^*-triple systems.

We therefore look in more detail at the constructions involved in (iv) and (v) in Theorem 4·4·3, when A is an embedable JB^*-algebra. As usual o denotes the Jordan product on A, and $\{\ ,\ ,\ \}$ denotes the normal Jordan triple product. If e is a tripotent in A, and we define a new product $\wedge : A \times A \to A$ by

$$x \wedge y = \{x, e^*, y\}$$

for x and y in A, it is clear that $(A, \wedge)$ is a commutative algebra.

Further, for all x and y in A, we have

$$\begin{aligned}(x \wedge x) \wedge (x \wedge y) &= \{\{x,e^*,x\},e^*,\{x,e^*,y\}\}\\ &= \{x,e^*,\{\{x,e^*,x\},e^*,y\}\}\\ &= x \wedge ((x \wedge x) \wedge y)\end{aligned}$$

by Macdonald's theorem. Hence $(A, \wedge)$ is a Jordan algebra, e is an idempotent in $(A, \wedge)$ and, by Theorem 2·3·7, $\|x \wedge y\| \leq \|x\|\|y\|$. Thus $(A, \wedge)$ is a complex Banach Jordan algebra.

THEOREM 4·4·4. Let A be an embedable JB*-algebra, and let e be a tripotent in A. Let $\wedge$ denote the new Jordan algebra product constructed above. If $S = \{x : \{e,x^*,e\} = x\}$, then S is a closed real Jordan subalgebra of $(A, \wedge)$, and $S \oplus iS = \{x : x \wedge e = x\} = \{x : \{e,\{e^*,x,e^*\},e\} = x\}$. Moreover, if $\dagger$ is the natural involution on $S \oplus iS$, then $x^\dagger = \{e,x^*,e\}$ for x in $S \oplus iS$, and $S \oplus iS$, with product $\wedge$ and involution $\dagger$ is a unital JB*-algebra.

<u>Proof</u>. It is clear that S is a closed real linear subspace of A. Given x and y in S, it follows by Macdonald's theorem that

$$\begin{aligned}U_e((x \wedge y)^*) &= \{e,\{x^*,e,y^*\},e\}\\ &= \{e,\{x^*,\{e,e^*,e\},y^*\},e\}\\ &= \{\{e,x^*,e\},e^*,\{e,y^*,e\}\}\\ &= x \wedge y .\end{aligned}$$

Hence $x \wedge y \in S$.

If $y \in S \cap iS$, then $i\{e,y^*,e\} = iy = -i\{e,y^*,e\}$, and so $y = 0$.

Hence, $S\oplus iS$ is a complex Jordan subalgebra of $(A,\wedge)$.

Suppose $a,b \in S$. Then

$$\begin{aligned}\{e,e^*,a\} &= \{e,e^*,\{e,a^*,e\}\}\\ &= \{\{e,e^*,e\},a^*,e\}\\ &= \{e,a^*,e\}\\ &= a\end{aligned}$$

by Macdonald's theorem, and so $\{e,e^*,a+ib\} = a + ib$.

If $z \in A$ satisfies $\{e,e^*,z\} = z$, then, by Macdonald's theorem

$$\begin{aligned}\{e,\{e^*,z,e^*\},e\} &= 2\{e,e^*,\{e,e^*,z\}\}-\{\{e,e^*,e\},e^*,z\}\\ &= z .\end{aligned}$$

Now suppose $w \in A$ satisfies $\{e,\{e^*,w,e^*\},e\} = w$. Let $p = \frac{1}{2}(w+\{e,w^*,e\})$ and $q = \frac{1}{2}i(\{e,w^*,e\} - w)$. Then $\{e,p^*,e\} = p$ and $\{e,q^*,e\} = q$, so that $w = p + iq \in S\oplus iS$.

Thus $S\oplus iS = \{x:x\wedge e=x\} = \{x:\{e,\{e^*,x,e^*\},e\} =x\}$, so that $S\oplus iS$ is a closed Jordan algebra with unit e. Also, if $w \in S\oplus iS$ and $p = \frac{1}{2}(w+\{e,w^*,e\})$, then $p \in S$, and so $w^\dagger = \{e,w^*,e\}$.

Finally, if $[\ ,\ ,\ ]$ denotes the triple product in the Jordan algebra $(A,\wedge)$, for all x in $S\oplus iS$, we have by Macdonald's theorem,

$$\begin{aligned}[x,x^\dagger,x] &= 2(x\wedge x^\dagger)\wedge x - (x\wedge x)\wedge x^\dagger\\ &= 2\{\{x,e^*,x^\dagger\},e^*,x\} - \{\{x,e^*,x\},e^*,x^\dagger\}\\ &= \{x,\{e^*,x^\dagger,e^*\},x\}\\ &= \{x,\{e^*,\{e,x^*,e\},e^*\},x\}\\ &= \{x,(\{e,\{e^*,x,e^*\},e\})^*,x\}\\ &= \{x,x^*,x\} .\end{aligned}$$

Hence $\|[x,x^\dagger,x]\| = \|x\|^3$, and so $S\oplus iS$ is a unital JB^*-algebra.

COROLLARY 4·4·5. The following are equivalent ;

(i) $S \oplus iS = A$,

(ii) A has a unit $e \circ e^*$,

(iii) e is a unit for $(A, \wedge)$.

Proof. (We have kept the notation introduced in Theorem 4·4·4).

(i) ⇒ (ii) . By Theorem 4·4·4, $\{e, e^*, x\} = x$ for all x in A . By Theorem 3·3·6, $\{e, e^*, x\} = x$ for all x in A''. In particular, as A'' has a unit, $e \circ e^* = 1$. Hence $e \circ e^*$ is a unit for A .

(ii) ⇒ (i) . By Lemma 4·1·1, e is unitary, and so $x = \{e, \{e^*, x, e^*\}, e\}$ for all x in A . Hence $A = S \oplus iS$ by Theorem 4·4·4.

(i) ⇔ (iii). As $S \oplus iS = \{x : x \wedge e = x\}$, e is a unit for A if and only if $A = S \oplus iS$.

If Y is a unital JB*-algebra, and u is a unitary in Y , then Theorem 4·4·4 and Corollary 4·4·5 allow us to construct a different Jordan algebra structure and involution on Y such that

$$\{x, x^*, x\} = [x, x^\dagger, x]$$

for all x in Y . Moreover, if $u^* = u$, then

$$(x^\dagger)^* = \{u, x^*, u\}^* = \{u, x, u\} = (x^*)^\dagger$$

for all x in Y , while if u is not central, $x^\dagger \neq x^*$ in general. We contrast this result with Theorem 4·1·9.

COROLLARY 4·4·6. Let Y be a unital JB*-algebra, with Jordan product $\circ$, unit 1, and involution $*$. Suppose that Y , with the same Banach space structure, is also a unital JB*-algebra under a second algebra product $\wedge$, with unit e and involution $\dagger$. Then $e \circ e^* = 1$ and

$$\{x, x^*, x\} = [x, x^\dagger, x]$$

for all x in A .

Proof If I is the identity map of $(A, \wedge, {}^{\dagger})$ onto $(A, o, {}^{*})$ then by Theorem 4·3·5, $e \circ e^{*} = 1$. The result now follows by the remarks before the Corollary.

Finally, we note that Theorem 4·3·1, which showed that the open unit ball of a unital JB*-algebra is biholomorphically equivalent to the "upper half-plane", generalises to some other B*-triple systems, provided "upper half-plane" is replaced by Siegel or tube domain; see [51]. These include all finite dimensional B*-triple systems [51], all C*-triple systems with a partial isometry [40], and all unital JB*-algebras [19] or Theorem 4·3·1. This, however, is not always true, as the open unit ball of a C*-algebra is biholomorphically equivalent to a Siegel domain or a tube domain if and only if A has a unit [52].

CHAPTER 5

In this chapter, we present some conjectures and open problems in the theory of Banach Jordan algebras, and JB*-algebras in particular. Seven different topics are covered: compact JB*-algebras, an extension of the Shirshov Cohn theorem, embedable JB*-algebras, an extension of Kadison's Schwarz inequality [47] to JB-algebras, a type decomposition for JB*-algebras, Banach space properties of JB*-algebras, and derivations on JB*-algebras.

A) Compact JB*-algebras.

DEFINITION. Let A be a complex Banach Jordan algebra. A is called compact if, for all a in A, U_a is a compact linear operator.

The definition of a compact Banach algebra was introduced by Alexander in [2], and the above definition is the analogue for Banach Jordan algebras. It is easy to see that [14] 33·11, 33·12 and 33·13 generalise to compact Banach Jordan algebras. The following is a slightly more substantial result.

THEOREM 5·1. Let A be a compact Banach Jordan algebra, and let $a \in A$. Then, for all $\eta \in \mathbb{R}^+ \setminus \{0\}$, $\sigma(a) \cap \{z \in \mathbb{C} : |z| \geq \eta\}$ is finite, and $\sigma(a)$ is countable. Moreover, for all $\lambda \in \sigma(a^2) \setminus \{0\}$, there exists an idempotent e in $P(a^2)$ such that

$$\{b \in P(a^2) : (a^2-\lambda)^n b = 0\} = eP(a^2) ,$$

where n is the ascent of the eigenvalue λ.

Proof. By [14] Lemma 33·6, as $P(a)$ is a compact Banach algebra, and $U_b|_{P(a)} = L_{b^2}|_{P(a)}$ for all b in $P(a)$, for all $\eta \in \mathbb{R}^+ \setminus \{0\}$

$\sigma_{P(a)}(a^2) \cap \{z \in \mathbb{C} : |z| \geq \eta\}$ is finite. Thus $\sigma(a) \cap \{z \in \mathbb{C} : |z| \geq \eta\}$ is finite, and so $\sigma(a)$ is countable. The final statement follows from [14] Theorem 33·8.

COROLLARY 5·2. Let A be a compact Banach Jordan algebra, and let $a \in A$. If $r(a) > 0$, there exists an idempotent e in $P(a)$ such that $U_e A$ is finite dimensional.

Proof. By Theorem 5·1, there exists a non-zero idempotent $e \in P(a^2) \subseteq P(a)$. e is a unit for the Banach Jordan algebra $J = U_e(A)$, so $U_e|_J$ is the identity operator on J. Hence, as J is compact, J is finite dimensional.

COROLLARY 5·3. Let A be a compact JB*-algebra, and let $a \in A$. Given $\eta \in \mathbb{R}^+ \setminus \{0\}$, there exists a finite set of projections $\{e_j\}$ $(1 \leq j \leq n)$ and a finite set $\{\lambda_j\}$ in $\mathbb{C}$ $(1 \leq j \leq n)$ such that

$$\|a - \sum_{j=1}^{n} \lambda_j e_j\| < \eta$$

and $U_{e_j} A = \mathbb{C} e_j$ $(1 \leq j \leq n)$.

Proof. $a = a_1^2 - a_2^2 + ia_3^2 - ia_4^2$ where a_1, a_2, a_3 and a_4 are self-adjoint and positive elements of $\|a\|A_1$. By Theorem 5·1, $K = \sigma(a_1) \cap \{z \in \mathbb{C} : |z| \geq \frac{1}{4}\eta\}$ is finite, and we let $K = \{\mu_1, \ldots, \mu_m\}$. For each μ_k, there exists an idempotent p_k in $P(a_1)$ such that

$$\|a_1^2 - \sum_{k=1}^{m} \mu_k p_k\| < \tfrac{1}{4}\eta.$$

As in the proof of Corollary 5·2, $U_{p_k} A$ is finite dimensional, and hence $p_k = \sum_{j=1}^{s_k} e_j$ where $\{e_j, A, e_j\} = \mathbb{C} e_j$ by [43] p158 and p205.

Thus,

$$\|a_1^2 - \sum_{k=1}^{m} \mu_k \sum_{j=1}^{s_k} e_j\| < \tfrac{1}{4}\eta .$$

A similar result holds for a_2^2, a_3^2 and a_4^2, and the general result now follows easily.

Even with such a powerful structure theorem as Corollary 5·3, it does not seem possible to produce a representation theorem, without using the methods of Alfsen Shultz and Størmer in [5]. However, some simplification is possible, as, if A is a compact embedable JB^*-algebra, each projection e_j constructed in Corollary 5·3 is a minimal projection in $A \oplus \mathbb{C}$, and also in the envelopping algebra of $Her(A \oplus \mathbb{C})$. Hence, only "type I" representations of $(Her(A \oplus \mathbb{C}))^\sim$ need be considered.

LEMMA 5·4. Every compact closed self-adjoint subalgebra E of the complexification of a type I_n JB-factor is finite dimensional if $n < \infty$.

Proof. If $3 \leq n < \infty$, it follows from [5] Proposition 8·3, that E is finite dimensional. A I_2 JB-factor is a spin factor by [5] Proposition 7·1. Hence if F is a closed subalgebra of a I_2 factor, then F is a Hilbert space in an equivalent norm [85], and in particular is reflexive. Hence by [78], Theorem 3·7, F has a unit. Hence, E has a unit, and so, as it is compact, E is finite dimensional.

In [80], Størmer classified type I_∞ factors. This leads to the following conjecture; for the notion of a quaternionic Hilbert space, we refer to [89].

CONJECTURE. Let A be an embedable compact JB^*-algebra. Then A is isometrically *-isomorphic to $\Sigma_o\{A_\lambda : \lambda \in \Lambda\}$, where each A_λ is the complexification of the image of the compact self-adjoint elements of $B(\mathcal{H})$ where $\mathcal{H}$ is a real, complex or quaternionic Hilbert space under a Jordan homomorphism, or $A_\lambda = M_3^8 \oplus i M_3^8$, and

$$\Sigma_o\{A_\lambda : \lambda \in \Lambda\} = \{f \in \Sigma A_\lambda : \forall \eta \in \mathbb{R}^+ \setminus \{0\}, \{\lambda : \|f(\lambda)\| > \eta\} \text{ is finite}\} .$$

B) An extension of the Shirshov-Cohn Theorem.

Our work on JB*-algebras has largely depended on Corollary 1·3·12, the extension of the Shirshov-Cohn theorem for unital JB-algebras. This raises the following problem, which we conjecture is false in general.

PROBLEM. Let A be a unital Banach Jordan algebra. Given a and b in A, is $P(1,a,b)$ special?

This is true if A is an alternative algebra and (A,o) is a unital Banach Jordan algebra, where $a \circ b = \frac{1}{2}(ab+ba)$ for a and b in A by [70], Theorem 3·1. We now show it is also true for unital JB*-algebras.

THEOREM 5 5. Let A be a unital JB*-algebra, and let $a, b \in A$. Then $P(1,a,b)$ is special.

Proof By Theorems 2·4·8 and 2·4·9, $P(1,a,b,a^*,b^*)$ may be embedded as a JB*-subalgebra of $B(\mathcal{H}) \oplus \Sigma\{M_3^8 \oplus iM_3^8 : f \in J\}$. If $f \in J$, then $f(P(1,a,b)) = f(Q(1,a,b))$ as f is continuous, and $\mathrm{Im} f$ is finite dimensional. Thus $f(P(1,a,b))$ is special, and so there is a one to one Jordan homomorphism of $f(P(1,a,b))$ into an associative algebra S_f . Thus $P(1,a,b)$ may be embedded into the associative algebra $B(J)\oplus\Sigma\{S_f : f \in J\}$.

We remark that an approach to the problem involving free special Jordan algebras or tensor algebras is not suited to the theory of Banach Jordan algebras. A related problem, which we also conjecture is false in general is the following.

PROBLEM. If A is a special unital Banach Jordan algebra, is A homeomorphically isomorphic to a closed Jordan subalgebra of a unital Banach algebra?

C) Alternative hypothesis under which a unit may be adjoined to a JB^*-algebra.

We noted in passing in Chapter 3, that the method used for adjoining a unit to a B^*-algebra does not work for JB^*-algebras without some modification. Here, we sketch some of the possible modifications.

THEOREM 5·6. Let A be a JB^*-algebra which does not have a unit, J the Banach Jordan algebra obtained by adjoining a unit, B the set of self-adjoint elements of A , and K the set of self-adjoint elements of J . Suppose that, for all x and y in B and all d in K ,

(i) $\|\{x,y,x\}\| \leq \|x\|^2 \|y\|$;

(ii) $\sup\{\|\{d,z,d\}\| : z \in B_1\} = (\sup\{\|\{z,d,z\}\| : z \in B_1\})^2$.

Then, for all d and c in K , we have $r(d+c) \leq r(d) + r(c)$.

Proof. Define $\mu : K \to \mathbb{R}^+$ by

$$\mu(d) = (\sup\{\|\{d,z,d\}\| : z \in B_1\})^{\frac{1}{2}} = \sup\{\|\{z,d,z\}\| : z \in B_1\} .$$

It follows easily that μ is a norm on K such that $\mu(b) = \|b\|$ for b in B, and $\mu(d) \leq \|d\|$ for all d in K. As $K = B \oplus \mathbb{R}$, and (B,μ) is a Banach space, (K,μ) is a Banach space and so μ is an equivalent norm to $\|.\|$.

Let $d \in K$, and let $h, k \in P(1,d) \cap K$. Then there exists $\{h_n\},\{k_n\} \subseteq Q(1,d) \cap K$ such that $h_n \to h$ and $k_n \to k$. By Macdonald's theorem, as $h_n, k_n \in Q(1,d)$, for all z in B we have

$$\{h_n,\{k_n,z,k_n\},h_n\} = \{h_n \circ k_n, z, k_n \circ h_n\} .$$

Thus, either $\{k_n,z,k_n\} = 0$, in which case $\{h_n \circ k_n, z, h_n \circ k_n\} = 0$,

or $$\|\{h_n \circ k_n, z, h_n \circ k_n\}\| \leq \|\{h_n,x,h_n\}\| \ \|\{k_n,z,k_n\}\| ,$$

where $x = \|\{k_n,z,k_n\}\|^{-1}\{k_n,z,k_n\}$. Hence, for all $n \in \mathbb{P}$, we have

$$\mu(h_n \circ k_n) \leq \mu(h_n)\mu(k_n) .$$

As $(K,\circ,\|.\|)$ is a Banach Jordan algebra, and μ is an equivalent norm on K,

$$\mu(h \circ k) = \lim \mu(h_n \circ k_n) \leq \lim \mu(h_n) \lim \mu(k_n) = \mu(h)\mu(k) .$$

Thus, (K,μ) is a real Banach Jordan algebra.

If $y \in B_1$, either $\{y,h^2,y\} = 0$, in which case $\{h,y,h\} = 0$,

or $$\begin{aligned} \|\{h,y,h\}\|^2 &= \|\{h,y,h\}^2\| \\ &= \|\{h,\{y,h^2,y\},h\}\| \\ &= \|\{h,w,h\}\| \ \|\{y,h^2,y\}\| , \end{aligned}$$

where $w = \|\{y,h^2,y\}\|^{-1}\{y,h^2,y\}$. So

$$\|\{h,y,h\}\|^2 \leq (\mu(h))^2\mu(h^2) ,$$

and hence $(\mu(h))^4 \leq (\mu(h))^2\mu(h^2)$. Thus,

$$(\mu(h))^2 \leq \mu(h^2) .$$

In particular, $\mu(d^2) = (\mu(d))^2$, so that

$$\mu(d) = r(d) .$$

Hence, for all c and d in K , we have

$$r(c+d) \leq r(c) + r(d) .$$

If A is a non-unital JB*-algebra, which satisfies the hypothesis of Theorem 5·6, it follows by Theorem 3·1·12 and Lemma 3·2·1 that A is embedable. We note that condition (i) is satisfied in a Banach algebra, while (ii) is a replacement for the left regular representation being an algebra homomorphism. An example of an occasion when (ii) holds is given in the following Theorem. The proof is not very different from that in [90] for B*-algebras, and we shall omit it.

THEOREM 5·7. Let A be a non-unital JB*-algebra, B the self-adjoint elements of A , and K the Banach Jordan algebra obtained by adjoining a unit to B. Suppose that ,

(i) $\|\{x,y,x\}\| \leq \|x\|^2\|y\|$ for all x and y in A ;

(ii) there exists a net $\{e_\lambda\}$ of positive elements in B_1 such that $\{e_\lambda\}$ is increasing, and, for all x in B , $\{e_\lambda,x,e_\lambda\} \to x$.

Then, for all d in K ,

$$\sup\{\|\{y,d,y\}\| : y \in B_1\} = \lim\|\{e_\lambda,d,e_\lambda\}\|$$
$$= (\lim\|\{d,e_\lambda,d\}\|)^{\frac{1}{2}} = (\sup\{\|\{d,y,d\}\| : y \in B_1\})^{\frac{1}{2}} .$$

D) Kadison's Schwarz Inequality.

If A is a unital C*-algebra and $\phi : A \to A$ is a continuous positive linear map such that $\phi(1) = 1 = \|\phi\|$, then a well known result of Kadison [47] states that, for all a in HerA ,

$$\phi(a^2) \gtrsim (\phi(a))^2 .$$

An easy proof, depending on Stinespring's Theorem [79], of this result is given in [82]. Positive, and completely positive linear maps between C^*-algebras have been very extensively studied recently, in mathematics and mathematical physics; see for example [82] and [92] and their references. As Kadison's Schwarz inequality does not hold for certain Segal systems without the distributive law, [75], it is natural to consider if the result extends to JB^*-algebras. We were unable to solve this.

PROBLEM. Let A be a unital JB^*-algebra, and let $\phi : A \to A$ be a continuous positive linear map such that $\phi(1) = 1 = \|\phi\|$. Is

$$\phi(a^2) \geq (\phi(a))^2$$

for all a in HerA ?

E) Type Theory.

A fundamental development in the theory of von Neumann algebras was the decomposition of a von Neumann algebra into Types; I_n , I_∞ , II_1 , II_∞ and III , where $n \in \mathbb{N}$. Topping [84] showed that a similar classification was possible for JW-algebras, and recently, Chu and Wright [21] have given a theory of type for the real ordered Banach spaces with order unit, and order unit norm, which are Banach dual spaces. Here, we outline the proofs of the results to show that the two classifications of [84] and [21] agree for JW-algebras.

Let A be a JW-algebra. A P-projection on A is a map U_p where p is an idempotent in A ([3] Theorem 12·13). A symmetry on A is an isometric Jordan automorphism ϕ of A onto itself such that

$\phi(a) \gtrsim 0$ if and only if $a \gtrsim 0$, $\phi(e) = e$, and $\phi L_a = L_a\phi$ whenever $a \in Z(A)$, the centre of A (by Theorem 3·4·3 and [95], Theorem 4). A central trace on A is a positive linear operator $\Gamma : A \to Z(A)$ such that $\Gamma(e) = e$, $\Gamma L_z = L_z\Gamma$ for all z in $Z(A)$, and $\Gamma\phi = \Gamma$ for all symmetries ϕ of A .

THEOREM 5·8. Let A be a JW-algebra. A has a central trace if and only if, there is a positive linear map $\psi : A \to Z(A)$ such that

(i) $\psi(e) = e$;

(ii) $\psi(za) = z\psi(a)$ for all $a \in A$ and all $z \in Z(A)$;

(iii) $\psi U_s a = \psi(a)$ for all $a \in A$ and all self-adjoint unitaries s in A .

Proof. Suppose Γ is a central trace on A . Then Γ is a positive linear map such that $\Gamma(e) = e$ and

$$\Gamma(za) = \Gamma(L_z a) = L_z\Gamma(a) = z\Gamma(a)$$

for all $a \in A$ and all $z \in Z(A)$. Moreover, if s is a self-adjoint unitary in A , then U_s is a symmetry on A , and so $\Gamma U_s(a) = \Gamma(a)$ for all a in A .

Conversely, suppose ψ is a positive linear map from A onto $Z(A)$ satisfying (i), (ii) and (iii) . Then

$$\psi L_z(a) = \psi(za) = z\psi(a) = L_z\psi(a)$$

for all $z \in Z(A)$ and $a \in A$. Let ϕ be a symmetry, and let $\lambda = \psi\phi$. Then, λ is a positive linear map of A onto $Z(A)$ such that $\lambda(e) = e$, $\lambda(za) = z\lambda(a)$, and $\lambda U_s(a) = \lambda(a)$ for all $a \in A$, all $z \in Z(A)$ and all self-adjoint unitaries s in A . Hence, by [84] Corollary 28, we have $\psi = \lambda = \psi\phi$. Hence ψ is a central trace on A .

By [84] Theorem 26 and Theorem 5·8, it follows that a

JW-algebra is modular if and only if it has a central trace. The argument of [21] p503 may be modified to show that a P-projection U_p is abelian if and only if p is an abelian projection [84]. A routine argument then shows that the decompositions of [21] and [84] agree for JW-algebra.

F) Banach space properties of JB^*-algebras.

As JB^*-algebras have less algebraic structure than B^*-algebras, it follows that Banach space theory is relatively more important in the study of JB^*-algebras, as we saw in Section 4·3. Here, we mention two results in the Banach space theory of B^*-algebras, which we feel will have interesting analogues for JB^*-algebras and more general order unit spaces.

In [38], Hamana gives a characterisation of those von Neumann algebras whose dual and/or predual satisfy the Dunford Pettis and/or some stronger properties. (The definition of the Dunford Pettis property is in [36].) As these are preserved by linear homeomorphism, it follows that, for example, if A is a von Neumann algebra all of whose irreducible representations are finite dimensional and of bounded degree, and B is a von Neumann algebra linearly homeomorphic to A, then the irreducible representations of B are finite dimensional and of bounded degree.

The second result is due to Pisier [63]. It is a Corollary of a more general Theorem, and was conjectured by Ringrose in [65], where equivalent formulations of the problem are given.

THEOREM 5·9. (Pisier). Let A, D be two C^*-algebras, and let

$u \in B(A,D)$. Then there exists a constant $\kappa (\leqslant 6^{\frac{1}{2}})$ such that, for all $n \in \mathbb{N}$ and all $x_1,\ldots,x_n \in A$,

$$\|(\sum_{j=1}^{n} u(x_j)o(u(x_j))^*)^{\frac{1}{2}}\| \leqslant \kappa\|u\| \ \|\sum_{j=1}^{n} x_j o x_j^*\|^{\frac{1}{2}} .$$

We conjecture that this Theorem also holds for unital JB^*-algebras.

G) Derivations of JB^*-algebras.

An important theorem on von Neumann algebras ([69] Theorem 4·1·6) states that every *-derivation is inner, that is, if M is a von Newmann algebra, and δ is an associative *-derivation on M, then there exists $a \in \text{Her}M$ such that $\delta(x) = ax - xa$ for all x in M. Even to state a possible analogue of this result, we require a suitable generalisation of the notion of inner derivations.

DEFINITION. If A is a JB^*-algebra, an inner derivation on A is an element in the closure of the set $\mathcal{E}(A)=\{\sum_{j=1}^{n}(L_{a_j}L_{b_j}-L_{b_j}L_{a_j}):n \in \mathbb{N}, a_j, b_j \in A\}$.

It is easy to check that each element of $\mathcal{E}(a)$ is a derivation on A ([43],p35). In addition, every derivation on a finite dimensional JB^*-algebra is inner by [43], p324. We now show that a Jordan *-derivation on a von Neumann algebra is inner in the sense of the above definition.

THEOREM 5·10. Let M be a von Neumann algebra, and let δ be a Jordan *-derivation on M. Then, given $\eta \in \mathbb{R}^+\setminus\{0\}$, there exists $n \in \mathbb{P}$, $a_1,\ldots,a_n$, $b_1,\ldots, b_n \in M$, such that, for all x in M,

$$\|\delta(x) - \sum_{j=1}^{n} (L_{a_j}L_{b_j} - L_{b_j}L_{a_j})\| < \eta\|x\| .$$

Proof. By Theorem 4·2·2 and [77] Theorem 3·3, δ is an associative $*$-derivation. Hence, by [69], Theorem $\bar{4}$·1·6, there exists $y \in \mathrm{Her} M$ such that $\delta(x) = yx - xy$. $M = F \oplus I$, where F is a finite von Neumann algebra and I is an infinite von Neumann algebra. Let $y = u + w$ where $u \in F$ and $w \in I$. By [83] Theorems 2 and 3, there exists $n, m \in \mathbb{P}$, $a_j, b_j, c_k, d_k \in M$ such that

$$w = \sum_{j=1}^{n} (a_j b_j - b_j a_j) \text{ and } \|u - \sum_{k=1}^{m} (c_k d_k - d_k c_k) - z\| < \tfrac{1}{2}\eta,$$

where z is the trace of u in F, and so z is in the centre of M. So

$$\|\delta(x) - \sum_{j=1}^{n} (L_{a_j} L_{b_j} - L_{b_j} L_{a_j})(x) - \sum_{k=1}^{m} (L_{c_k} L_{d_k} - L_{d_k} L_{c_k})(x)\|$$

$$\leq 2\|w - \sum_{j=1}^{n} (a_j b_j - b_j a_j)\| \, \|x\| + 2\|u - \sum_{k=1}^{m} (c_k d_k - d_k c_k) - z\| \, \|x\|$$

$$< \eta \|x\| .$$

I am indebted to Mr. T.R.Behrndt for pointing out Sunouchi's results in [83], and for giving a proof of the following Theorem. If A is a JW-algebra, and $C, B \subseteq A$, we let

$$[B,C] = \{\sum_{j=1}^{n} [b_j, c_j] : n \in \mathbb{N},\ b_j \in B \text{ and } c_j \in C\}.$$

THEOREM 5·11. Let A be a JW-algebra with a faithful normal central trace ψ. If B is the norm closure of $[[A,A],A]$, then $B + Z(A) = A$.

Proof. By [84] Lemma 31 and Theorem 25, if $a \in A$ and $\eta \in \mathbb{R}^+ \setminus \{0\}$, there exist $\{\lambda_j\} \subseteq \mathbb{R}^+$ such that $\sum_{j=1}^{n} \lambda_j = 1$, and self-adjoint unitaries $s_{1,j}, \ldots, s_{m,j} \in A$ such that

$$\|\psi(a) - \sum_{j=1}^{n} \lambda_j s_{m,j} s_{(m-1),j} \cdots s_{1,j} a s_{1,j} \cdots s_{(m-1),j} s_{1,j}\| < \eta .$$

It follows by induction that

$$s_{m,j}\cdots s_{1,j}as_{1,j}\cdots s_{m,j} - a = \tfrac{1}{2}([[s_{m,j},s_{m-1,j}\cdots s_{1,j}as_{1,j}\cdots s_{m-1,j}],s_{m,j}] + \cdots+[[s_{2,j},s_{1,j}as_{1,j}],s_{2,j}]+[[s_{1,j},a],s_{1,j}]) .$$

Thus

$$\|\psi(a)-a-\tfrac{1}{2}\sum_{j=1}^{n}\lambda_j([[s_{m,j},s_{m-1,j}\cdots s_{1,j}as_{1,j}\cdots s_{m-1,j}],s_{m,j}]+\cdots+[[s_{1,j},a],s_{1,j}])\|$$

$$= \|\psi(a) - \sum_{j=1}^{n}\lambda_j s_{m,j}\cdots s_{1,j}as_{1,j}\cdots s_{m,j}\|$$

$$< \eta .$$

So $\psi(a) - a \in B$. However, $\psi(a) \in Z$, so that $a \in B + Z$. Hence $A \subseteq B + Z$. It is clear that conversely $B + Z \subseteq A$, and so $A = B + Z$.

Even with the above result we were not able to show that every derivation on a unital JB^*-algebra is inner. Hence, we conclude with the following two problems.

PROBLEM. Let A be a unital JB^*-algebra which is a Banach dual space. If $[[A,A],A]$ denotes the linear span of the elements of the form $(\{a,b,c\} - \{b,a,c\})$ for a, b and c in A , and B is the norm closure of $[[A,A],A]$, is $B + \mathcal{Z}(A) = A$?

PROBLEM. If A be a unital JB^*-algebra which is a Banach dual space, is every derivation on A inner?

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